КВАНТОВЫЕ СТРУКТУРЫ И КВАНТОВОЕ МОДЕЛИРОВАНИЕ

QUANTUM AND MOLECULAR COMPUTING AND QUANTUM SIMULATIONS

Conference “Quantum Informatics – 2021”

Organized by the Faculty of Computational Mathematics and Cybernetics of the Lomonosov Moscow State University with the support of the Moscow Center for Fundamental and Applied Mathematics.

Conference was hold in March-April of 2021. It focused on computational aspects of quantum mechanics, quantum computers, and quantum communications. Over the past 10 years, quantum computer science has become the most important scientific field that determines progress in micro- and nano-electronics, biotechnologies, complex chemistry, and information security. The special role of quantum cryptographic protocols for protecting information when it is transmitted over communication lines has been proven by the practice of using such protocols in many countries. The quantum computer project, which is being developed in the world’s leading centers, is of fundamental importance for science as a whole; for example, it should enable the management of vital processes for both the individual and society. The quantum computer science tool – computer and supercomputer computations and modeling of complex processes at the quantum level – puts the faculty of Computational mathematics and Cybernetics of MSU – VMK, its mathematicians and programmers, in a leading position in this direction.

The main goal of the conference is to consolidate the efforts of scientists working in the quantum field in the different centers of the world, to inform each other about the results of their work and to discuss future plans. This will increase the efficiency of research conducted at the VMK faculty and enhance the effectiveness of both traditional mathematical areas and the use of supercomputing and other super-productive computing methods in the most important applied areas. The development of research in quantum computer science will also give programmers new interesting and important tasks, for example, this applies to the operating system of a quantum computer and its fragments. The conference will strengthen the ties between the different groups in Russia and abroad, which deal with quantum topics, as well as improve the teaching of quantum mechanics.

Key topics of the conference:

- quantum computers, computing, quantum operating system, gates and their implementations;
- quantum cryptography and quantum information theory;
- modeling of quantum systems, solving the Schrödinger equation, direct and inverse problems of scattering of several particles
- Feynman diagrams;
- calculations and modeling of quantum devices: Lasers, photodetectors, quantum dots, superconducting elements;
- quantum elements and methods in supercomputing and distributed computing, the quantum side of Big Data and artificial intelligence;
- statistical methods of quantum theory, quantum random processes;
- algebraic methods of quantum computer science;
- quantum aspects of biology and biochemistry;
- quantum methods of management and decision-making;
- interdisciplinary applications of quantum mechanics, quantum economics and quantum politics;
- teaching quantum theory.

The main part of papers presented at the conference are published in the preprint archive http://arxiv.org, in the quant-ph section. Articles selected by the program Committee during the review process are also going to be published in journals “Nonlinear Phenomena in Complex Systems”, “Computational Mathematics and Modeling” and “Computational nanotechnology”.

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Conference program, abstracts of reports and video of the sessions are available from the links at the conference page: https://vql.cs.msu.ru/QI-2021eng.html

Конференция «Квантовая информатика – 2021»

Конференция «Квантовая информатика – 2021», организованная факультетом Вычислительной математики и кибернетики Московского государственного университета им. М.В. Ломоносова, при поддержке Московского центра фундаментальной и прикладной математики, состоялась в марте-апреле 2021 г.

Она была посвящена вычислительным аспектам квантовой механики, квантовым компьютерам и квантовым коммуникациям. Квантовая информатика в последние 10 лет стала важнейшим научным направлением, определяющим прогресс в микро- и наноэлектронике, биотехнологиях, сложной химии, и защите информации. Особая роль квантовых криптографических протоколов для защиты информации при ее передаче по линиям связи проверена практикой использования таких протоколов как в нашей стране, так и за рубежом. Проект квантового компьютера, разрабатываемый в ведущих мировых центрах, имеет принципиальное значение для науки в целом; например, он должен дать возможность управления жизненно важными процессами как для отдельного человека, так и для общества. Инструмент квантовой информатики – компьютерные и суперкомпьютерные вычисления и моделирование сложных процессов на квантовом уровне – выдвигают факультет Вычислительной математики и кибернетики, его математиков и программистов, на ведущие позиции в этом направлении.

Главная цель конференции – консолидация усилий российских ученых, работающих в квантовой области, взаимное информирование о результатах работы и обсуждение планов на будущее. Это позволит повысить эффективность ведущихся в России научных исследований и усилить результативность как традиционных математических направлений, так и применения суперкомпьютерных и иных сверхпроизводительных методов вычислений в важнейших прикладных областях. Развитие исследований по квантовой информатике даст также новые интересные и важные задачи программистам, например, это относится к операционной системе квантового компьютера и ее фрагментам. Конференция способствовала укреплению связей ученых России, занимающихся квантовой тематикой, а также усовершенствованию преподавания квантовой механики.

Важнейшие темы конференции:

- квантовые компьютеры, вычисления, квантовая операционная система, гейты и их реализации;
- квантовая криптография и квантовая теория информации;
- моделирование квантовых систем, решение уравнения Шредингера, прямой и обратной задач рассеяния нескольких частиц
- фейнмановские диаграммы;
- расчеты и моделирование работы квантовых приборов: лазеров, фотодетекторов, квантовых точек, сверхпроводящих элементов;
- квантовые элементы и методы в суперкомпьютерных и распределенных вычислениях, квантовая сторона Big Data и искусственного интеллекта;
- статистические методы квантовой теории, квантовые случайные процессы;
- алгебраические методы квантовой информатики;
- квантовые аспекты биологии и биохимии;
- квантовые методы управления и принятия решений;
- междисциплинарные приложения квантовой механики, квантовая экономика и квантовая политика;
- преподавание квантовой теории.

ПРОГРАММНЫЙ КОМИТЕТ КОНФЕРЕНЦИИ «КВАНТОВАЯ ИНФОРМАТИКА – 2021»

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Большинство работ, представленных на конференции, можно найти в архиве препринтов http://arxiv.org, в разделе quant-ph с аффилиацией QI-2021. Избранные программным комитетом в процессе рецензирования статьи приняты к публикации в журналах «Nonlinear Phenomena in Complex Systems», «Computational Mathematics and Modeling» и «Computational nanotechology» в соответствии с правилами этих журналов. Программа конференции, абстракты статей, а также видеозаписи сессий можно найти, следуя по ссылкам на сайте Виртуальной квантовой лаборатории факультета ВМК МГУ: https://vql.cs.msu.ru

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On the Physical Representation of Quantum Systems

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Abstract. The Schrödinger equation for bound states depends on a second derivative, that only exists if the solution is continuous, which is – by itself – contradictory, and cannot be digitally calculated. Photons can be created in-phase by stimulated emission or annihilated by spontaneous absorption, and break the LEM, more likely at lower frequencies, and even in vacuum. Thus, the number of particles is not conserved, e.g., in the double-slit experiment, even at low-light intensity. Physical representations of quantum computation (QC), cannot, thus, follow some customarily assumed aspects of quantum mechanics. This is solved by considering the Schrödinger equation depending on the curvature, which is expressed exactly as a difference equation, works for any wavelength, and is variationally solved for natural numbers, representing naturally the quantum energy levels. This leads to accepting both forms in a universality model. Further, one follows the Bohr model in QC, in a software-defined QC, where $GF(2^m)$ can be used with binary logic to implement in software Bohr's idea of "many states at once", without breaking the LEM, in the macro, without necessarily using special hardware (e.g. quantum annealing), or incurring in decoherence, designed with today's binary computers, even a cell phone.

Key words: bound states, qubit, qutrit, qudit, tri-state+, information, algebraic, quantum, classical, coherence

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1. INTRODUCTION

In trying to open the "black box" in the quantum state, with further analysis of the interaction process where the data can make a wider causal sense, we hope to better understand the limitations about quantum processes. The development of nanoelectronics devices, when nanoprocesses needs to involve quantum computing, also needs prediction of the structure of matter. But, if the second derivative is to be included in quantum mechanics (QM), even though the second derivative only exists if the solution is continuous, which is – by itself – contradictory [1]. The standard justification for using derivatives with the wave function Ψ, describing a discrete behavior on/off without even continuity, is that Ψ represents an average behavior, in the Bohr interpretation.

But this leaves out many behaviors that are not continuous, and they do not have to be continuous for ψ to be an average, even for a continuous function interpolating isolated data points. A number of contradictions then arise from the use of infinitesimal analysis in QM, in particular, for the core Schrodinger equation, the very applicability of which turns out also to be limited by non-compliance with the conservation of the number of particles. We offer ways to overcome these contradictions. In particular, on the basis of replacement of the Schrodinger equation by an exact difference scheme, which results in a curvature representation, Eq.(4), that does not use continuity and yet can reproduces all known behaviors. It then reveals a remarkable behavior of all energy levels E_n, that they all scale as

$$E_n = an + b + \frac{c}{n} + O\left(\frac{1}{n^2}\right),$$

where n is the quantum number. This allows scaling laws to be calculated, even for very complicated potentials, such as for Rydberg levels.

Besides, a continuous solution, even as the average Ψ, cannot be digitally calculated, so that any digital code must be seen necessarily as an approximation of some expected (although mythical) "analog", continuous code. These two contradictions, and more, directly impact quantum computing (QC) and has diverse manifestations. In particular, in the need for renormalization in quantum electrodynamics.

The second derivative is indeed included, e.g., in the expression of the one-dimensional Schrödinger equation for bound states in *QM* [2]:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = \left[E - V(x)\right]\psi(x), \tag{1}$$

where E is the energy and $V(x)$ is the potential, with the boundary conditions ψ(0) = ψ(∞) = 0.

Besides the continuity question (i.e, with 'lack of continuity' and 'code is not continuous'), Eq. (1) brings in the Law of the Excluded Middle (LEM) as a third contradiction, in 'breaking the LEM'.

The LEM is broken in the double-slit experiment – reported in [3], because one can't say which of the two slits a particle takes in the experiment. Any attempt to determine this, would need an interaction with the particle, which would lead to decoherence, and consequent loss of interference. This as was concluded in [3], and one must also consider the well-known particle creation and annihilation. Photons can be created in-phase by stimulated emission or annihilated by spontaneous absorption, more likely at lower frequencies, even in vacuum, as predicted by Einstein [4; 5]. Thus, which effect must depend on wavelength, this represents a fourth contradiction – 'number of particles is not conserved'.

E.g., in the double-slit experiment, this means that one cannot say that only one photon exists in the apparatus, contrary even to [6]. There, in a well-known book, by Dirk Bouwmeester, Arthur Ekert, and Anton Zeilinger (eds.), they say, in page 1, that "...the interference pattern can be collected one by one, that is, by having such a low intensity that only one particle interferes with itself".

Why this would break the LEM? Because we can't say which of the two slits a particle takes in the experiment. Any attempt to determine this, would need an interaction with the particle, which would lead to decoherence, that is, loss of interference.

This reasoning is correct, but relies on having only one particle in the experiment, which is questionable. One may indeed have such low intensity as to inject only one particle in the experiment, but the particle can multiply in-phase, indistinguishably, by stimulated emission, or be annihilated, by spontaneous absorption. This happens more likely at low frequencies, and can happen in vacuum, as the experiment has at least one wall with two slits. Thus, even though injected with 1 particle, one may have 0, 1, 2 or more particles inside. In these cases, the LEM is also broken, and the situation cannot be avoided. The number of photons is not conserved. The reference perhaps considers, naively, that the number of particles is constant. But the LEM is, nonetheless, broken. It is broken by stimulated emission, which can produce an extra particle, and broken by default, by not being able to tell which slit was used. There is no YES or NO answer possible, for each of the two slits. That the reference would "forget" about stimulated emission is, nonetheless, incorrect. The reader is advised, though, that the LEM is indeed broken anyway.

Now, it becomes more forceful, as breaking the LEM is further helped by photon multiplication, producing two or more photons out of one, in the same phase space, using stimulated emission. The LEM is broken without any doubleslit, by the very existence of a third, coherent state, as found by Einstein in 1917 [5; 6]. These two or more "identical" photons, as Einstein found out in the B coefficients, are more likely in lower frequency. Albeit, the external field can be theoretically calculated, as follows.

For light (i.e., a photon) interacting with a double-slit, the general external state as understood by Eq. (1) is given by Ψ. Here, Ψ is also *the coherent superposition* of the solutions Ψ_a and Ψ_b, where only slit a or slit b are open at the same time:

$$\Psi = \frac{1}{\sqrt{2}}\left(\Psi_a + \Psi_b\right). \qquad (2)$$

Thus, the behavior of systems described by the Niels Bohr interpretation of *QM* [7] does *not* reproduce classical physics in the limit of small quantum numbers, although it reproduces for high quantum numbers, being counter-intuitive [6] to our usual observed experience, in those small numbers (see Universality discussion). With the Copenhagen interpretation [8; 9] in lieu of the Bohr interpretation, this would contradict also the observation of a particle to have matter or charge, such as electrons, protons and neutrons, or subatomic, and consider a spurious special role by one arbitrary, subjective observer producing an objectively, important to all, solipsistic "collapse". This leads us to consider, instead, the model of spatial averages in the Bohr interpretation, of the quantum level probability density function, as representing the particle in classical physics, although the former four contradictions remain.

The reason the double-slit experiment is counter-intuitive is because it breaks the LEM.[1] One cannot split the photon at the double-slit experiment, notwithstanding Huygens and all classical considerations, such as the Maxwell equations (ME) in any form, even when represented by relativistic equations for the field strength tensor, with B and E using the same units [3].

On these considerations, some computational aspects of quantum mechanics are to be hopefully improved. This will lead us to consider an equivalent form of Eq. (1) in an extended algebra approach, depending on the curvature, and valid for any frequency, which can be expressed *exactly* as a first-order difference equation, and is variationally solved for natural numbers, representing naturally the quantum energy levels.

While many are considering a far-future and expensive hardware solution with quantum annealing for QC, this work on QC sees the noted four contradictions of:

- 'lack of continuity',
- 'code is not continuous',
- 'breaking the LEM', and
- 'number of particles is not conserved'.

These are four openings to consider a "new hypothesis" here, promising a new, coherent basis for QC.

A shorter version was used in the actual presentation, and is available online at [10].

2. UNIVERSALITY AND INFINITESIMALS

Although QM unquestionably requires discrete values and breaking the LEM, it has been presumably accurate when using continuity and LEM to calculate them. But both sides of an equation representing a physical relationship with discrete values and no LEM, such as $A = B$, QM must be kept discrete and breaking the LEM when the frame of reference changes and the so-called continuum condition is denied.

To change the reference frame, a well-known theorem of topology [11], which we call Topological Relationship (TR), says that a generic one-to-one mapping between spaces of different dimensionality must be discontinuous, in that a continuous path in one space must map into a broken path in the other. The consequences here are multiple, and this is being explored in [3] as well. Thus, the mathematical condition seems to imply the physical condition, and continuity (e.g., including all four contractions, with LEM), denied in one frame, is to be denied in all.

It would be desirable, therefore, to isolate those aspects of the current QM theory that involve such continuous quantities in Eq. (1), if they exist, and are subject to modification by a more satisfactory theory, from aspects that involve only discrete values that do not obey the LEM, and are thus relatively more trustworthy.

This is a candidate for such a more satisfactory theory of QM.

Historically, however, it is well-known that Eq. (1), can correctly describe the evolution of bound states of a quantum physical system for high frequencies, and also work in terms of quantum, discrete variables, which results would need to be preserved under the "new hypothesis". This item argues that this is possible under a difference in scale as universality,

[1] Without photon creation or annihilation, effects more likely at lower frequencies, as well known. The same interference pattern would be observed with just one particle at a time (e.g., one photon) in the experiment, so that the photon must interfere with itself in this case.

and one thus may not be even able to bother with finer details when only seen at a distance.

If the world is quantum or not, anyway Newton's calculus and real numbers lead to infinitesimals and infinities, which were shown by Brillouin [12] to be non physical. No one can physically push spacetime arbitrarily close to zero, without limit, or exclude particle creation or annihilation at low frequencies. Galois fields and a finite difference approach can be used, to build an alternative to conventional calculus, without infinitesimals or the limit concept, so that the conventional approach of [1] need not to be the only approach to analysis, for physicists, as we shown in Section 3 on the curvature representation. The derivative and integral formulas, however, remain the same. The new finite difference approach can be exactly accurate and yet there is always a space between integers, which represent different points in spacetime.

However, while physics shows that nothing is continuous in nature and, although non physical, one can keep using infinitesimals, continuity, and irrationals in mathematics, aka continuity. No one needs to change the traditional treatment [1] of analysis (i.e., calculus) or limits. This is because of TR, or Topological Relationship, a well-known theorem in topology where a generic one-to-one mapping between spaces of different dimensionality must be discontinuous. Therefore, a higher dimensional state can embed in a lower-dimensional state, as well-known in topology, projection, and physics, although it is subject to TR.

Thus, the universe can have singularities, be quantum at the core, and yet reality is the consequence of a continuous-looking universality as discussed in Section 2, where we observe this through what can only be an ever far-away reference frame [13; 14], where we observe this through what can only be an ever far-away reference frame. Universality as the reason for the "black box" in the quantum state. The details of the microscopic, even breaking the conservation of the number of particles, and the LEM, should not be so relevant to the macroscopic behavior and asserting the LEM, in *universality* [13; 14]. Can the same be affecting QC, and that is why we do not see a strong microscopic effect resulting from *a*, even though coherent, microscopic cause?

Different from the interaction with a double-slit, as seen above, when a photon interacts with matter as in [4], one needs to consider not just passing or not passing an aperture in a wall, but further consider the equilibrium of the photon field with the material in the wall, even in vacuum. In 1916–1917, Einstein took this latter case and [5; 6] famously argued that, in addition to the random processes of spontaneous absorption and spontaneous emission in Eq. (1), a third, new, and coherent process of stimulated emission must exist *microscopically* for physical bodies, as a result providing experimental evidence for the quantum, allowing photon creation in-phase[2] and annihilation.

Einstein contradicted, thus, the well-known Maxwell equations, and reproduced exactly the experimental studies of the thermal, statistical radiation of bodies in quantum communication, which provided the basis for the later invention of the laser (light amplification by simulated emission of radiation).[3] This is the so-called black-body radiation law, macroscopic, and even normal light from a candle, a lamp, or, a radio wave, have a stimulated emission component. This has been extended recently, as well-known, with collective effects, such as superradiance and superabsorption, into 5 states, but with no essentially new process.

This work's "new hypothesis" is introduced here, where one moves from the classical Shannon Boolean analogy of circuits with relays, valid for the LEM and a formless and classical "fluid" model of information, with a syntactic expression called 'bit', to a quantum tri-state+, where information is given by an abstract, algebraic approach with ternary object symmetry and extensions, modeled by $GF(3^n)$ and implementable as $GF(2^m)$ [3].

For interaction with matter, we found [4] that Einstein's "stimulated emission" provides coherence in universality, and applies not only to bodies that we must use to transmit and receive information, but also to how we communicate.

This is based on the topological projection [4] properties, as $GF(3^n) \Rightarrow GF(2^m)$, for suitable $m > n$, with $GF(3^n), GF(2^m) \in \mathbb{Z}/\mathbb{Z}p$, where p is a prime number, meaning that any three-valued logic system, breaking thus the LEM, can be represented (i.e, embed) in a binary logical system, obeying the LEM, although subjected to TR.

For interactions with matter, we are to apply $GF(3^n)$ in behavior, but use $GF(2^m)$ [4] for practical implementation, using binary logic. For the double-slit experiment, and accepting the natural processes of particle in-phase creation and annihilation, we model behavior through the binary decision process in two stages as $GF(2^2)$. We use the first binary tree, by taking measurements from the front of the apparatus. After the double-slit, we take measurements again, from the immediate side after the apertures, and use a second binary tree. This makes it possible to use standard binary logic in QC, and always obey the LEM in the aggregate. Finer cases, with particle in-phase creation and annihilation, can be analysed by further binary trees, by using $GF(2^m)$.

Also, as known [15–20], the exact representation of Eq. (4) can be easily, analytically calculated for common and any potentials, instead of masked in high frequency by WKB [2] or numerical methods. The scaling law reported [16] would not had been revealed by WKB or numerical methods.

Although calculus requires continuity for the existence of derivatives, that is based not only on the four operations of arithmetic but also on the definition of real numbers. It does not seem necessary to require continuity, in general, as Cauchy did in analysis in the field of real numbers. In the field of finite integers, $\mathbb{Z}/\mathbb{Z}p$, such as in Galois fields, calculus can be defined exactly, as well-known, while not requiring continuity.

Continuity can be thought here as a *collective construct* from universality, not as a result from the use of infinitesimals (as Cauchy thought) that, are revealed, do not exist, and no such example can be shown [12; 21]. Code is also not continuous, and cannot create infinitesimals. Continuity, though, happens collectively also using Galois fields, finite integers, and code – through a *collective construct* also viewed from universality.

NATURE IS QUANTUM IN THE MICRO, CONTINUOUS IN THE MACRO

Mathematics can reflect nature, as Computer Science does today with digital only – where binary calculations belong to the reality we see, not as mythical "approximations" of an ideal analogue signal that does not exist or can be made.

[2] Stimulated emission, so that a photon still only interacts with a photon in the same phase space, i.e., 'identical'.

[3] More than 55 000 laser-related patents have been granted in the United States.

However, for more than 50 years, as explained first by Brillouin, nothing was an infinitesimal to the physicist [12] and one cannot calculate or code it – the topology is digital, not continuous – thus, continuity has had no topological meaning (as well as no Cauchy epsilon-deltas) in the physical sense. Computers have no ”continuity chip” – it is all done by binary code. Even so-called ”floating point processors” such as the old coprocessor are using binary data in calculating mantissas and exponents. Operations performed by the coprocessor may be floating point arithmetic, graphics, signal processing, string processing, cryptography or I/O interfacing with peripheral devices. Yet, computers can emulate continuous results, and CDs can play apparently continuous music, any rhythm, better than analogue. Analogue and continuity are the approximation, while digital is the true result. Code is the exact result we see everywhere, while analogue has been, more and more, deprecated. This is now justified, in universality, but leads one to consider what may be an artificial renormalization in quantum field theories [22], and an artificial continuity in general relativity, (GR) all of which should be quantum.

This comment then, opens mathematics and physics to a new understanding (e.g., opposing Brillouin), as follows:

- one keeps infinitesimals, Cauchy results, and continuity, even though they are not observable and are not able to be constructed ever, and proceeds with these hypothesis as IF true; or,
- one takes the side of the opposite hypothesis, uses the algebraic approach with only finite integer fields and extension fields (e.g., Galois fields), that obey all four mathematical operations (+ – ×/), and pursues further its consequences, such as the spurious continuity requirements in GR, and in the Schrцdinger equation.

The latter is a universality solution we propose, also to overcome a “sacred cow” feeling on infinitesimals [21] or even on the double-slit experiment, that are not obeyed classically (i.e., with infinitesimals, as one divides a volume to reach the supposed infinitesimal, soon one passes molecules, atoms, and even particles) nor in quantum physics, (i.e., number of particles is not conserved; indistinguishable photons can be created in-phase, or annihilated).

On the particle view of nature, the latter view imposes natural limits also on the coherence of collective effects not only on the isolated particle itself, such, as e.g., in the length of the particle. The issue calls attention to the resulting universality in the first case, as leading to continuity in the macrocosm, although no particle is ever continuous, in the microcosm, and can be observed only through what can ever be a far-away reference frame. This is in spite of the noted four contradictions of ’lack of continuity’, ‘code is not continuous’, ‘breaking the LEM’, and ‘number of particles is not conserved’.

Thus, the concept of universality allows us to use nice, known formulas such as derivatives in Eq. (1), and also the curvature method of Eq. (3), when only discrete values should be used, without conflict.

3. CURVATURE REPRESENTATION OF BOUND STATES IN QUANTUM MECHANICS

The second-derivative in Eq. (1) can be represented *exactly* as a first-order difference equation, in any function spanned by linear combinations in the set U:

$$U = \{e^{-\alpha x}, xe^{-\alpha x}, x^2 e^{-\alpha x}\}, \qquad (3)$$

with a suitable $\alpha > 0$ [16; 19; 20; 23; 24], which already obeys the known boundary conditions of the Eg. (1). This motivates us to eliminate the second derivative in Eq. (1), eschewing the hypothesis of continuity (needed by the derivative $d^2\psi(x)/dx^2$). Then, Eq. (1) becomes an equivalent first-order difference equation, normalizing in atomic units,

$$\psi\left(x_{k-1}\right) = \frac{e}{\alpha_k^2}\left[E - \alpha_k^2 - V\left(x_k\right)\right]\psi\left(x_k\right), \qquad (4)$$

where the index $k = 1, 2, 3, \ldots, K$ refers to the partition of the coordinate space, as usual, α_k is a piece-wise variational parameter, and the boundary conditions are $\psi(0) = \psi(\infty) = 0$, where ∞ represents a large enough, finite, separation in spacetime [19]. Eq. (3) is hereafter called the *curvature representation* of Eq. (1), and is discrete.

Based on these potentials and the spectra of all the other potentials tested at high and low frequencies [20; 23], including the logarithmic, power-law, and square-root, we expect that with the *curvature representation* in Eq. (3) we can *exactly* represent Eq. (1), although not in vice-versa. Eq. (3) provides insights not reachable by Eq. (1), such as the scaling law in [16]. The usual functional dependence of the eigenvalues is to be obtained by using the quantum number for a generic potential $V(x)$, and we expect it to reproduce all the eigenvalues of the usual Eq. (1).

We conjecture that Eq. (3) is an *exact* discrete representation of Eq. (1), although not in vice-versa, and without using any implicit or explicit continuity. Universality was not important at the micro level either – it happens at what can only be an ever far-away reference frame.

4. UNIVERSAL ITY MODEL

In this paper, we hope to use a clear difference in the regimes of small versus high quantum numbers to improve significantly upon QC using physical systems or computers, and the QC interpretation of calculated physics as a consequence. Is code result to be considered part of reality? How about continuity? Universality answers both questions.

In this regard, we consider what can be called the “universality model” of QC, in two cases:

1) Without the Heisenberg Principle. Also called the Bohr model. The quantum particles have well-defined location and velocity, but we are just not able to know them precisely, as it happens at what can only be an ever faraway reference frame. Niels Bohr [7] described that a quantum particle does not exist in one state or another, but in all of its possible states at once.[4] Here, Eq. (1) or Eq. (3), can then be used to determine the probability density distributions for a particle location and velocity.
2) With the Heisenberg Principle. All quantum particle states co-exist but, as exemplified by the “Schrödinger’s cat” mental experiment, only becoming a well-defined location and velocity when collapsing in the macro, upon measurement or observation by an observer.

The main difference between the two views in the “universality models” is collapsing the wave function, which is not a matter in the Bohr model. Here, we find the first model, without the Heisenberg Principle, to be more useful.

Experimentally, the behavior of systems described by the first “universality model”, the Niels Bohr theory of QM, does not reproduce classical physics in the limit of small quantum numbers, although it reproduces for high quantum

[4] Not to be confused with the complementarity principle, also formulated by Bohr.

numbers, being often counter-intuitive [6] to our usual observed experience (see Introduction).

This, which may be surprising at first, can be clarified by examples from complex analysis [25]. The fact that the product of two negative numbers is a positive number, also seems surprising at first. In the debit model, where the negative number is a debit, how to explain that the product of two debits is a credit? However, in a complex number model, a negative number is a 180° degree rotation, so the product of two such numbers is a 360° rotation, positive therefore.

Results begin and end in real number theory, but have a path through the complex plane, which influences the result, but remains hidden.

As Edward Titchmarsh [26] observed, $\sqrt{-1}$ is a much simpler concept than $\sqrt{2}$, which is an irrational number, essentially unknowable.

There are certainly people that regard $\sqrt{2}$ as something perfectly obvious, but sneer at $\sqrt{-1}$. This is because they think they can visualize the former as something in physical space, but not the latter.

This investigation uses the complement: one can not really visualize $\sqrt{2}$, but one can visualize $\sqrt{-1}$, as a 90° rotation, and apply it in physics and engineering of real systems. One could do this using the real-line only, but one will benefit from the complex plane [25], as shown here.

Complex numbers are not part of the reality that we can measure, as we can only measure finite integers (e.g., Galois numbers) and their ratio.

Mutatis mutandis, quantum numbers offer a similar opportunity in QC. The original "universality model" of QC is due to Bohr [7], that a quantum particle does not exist in one state or another, but in all of its possible states at once. This "universality model" does not need to have an analogue in real systems, nor in language, nor even in our mathematics, nor code, nor that one can necessarily realize it in a physical system, as "quantum annealing", nor that can avoid decoherence.

In QM, the energy field must also have a discontinuous, discrete structure,[5] where only the mathematical nature is evident as a description of reality, while a physical description of "continuity" is to be denied.

According to Khrennikov [27], the role of a mathematical apparatus in a description of reality implies that there exist other pictures of reality where other number fields are used as basic elements of a mathematical description.

According to Ozhigov [21], the problem of scalability in quantum computation represents the old question of the description of the measurements and decoherence in QM.

The problem of scalability in QC is made worse when using the Copenhagen interpretation [8] in lieu of the Bohr interpretation [7], whereas the collapsing of the wave function is not a matter in the Bohr model.

According to Bohr [7], there is no quantum world: "*This is only an abstract physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature*". We see this as an universality view of nature, where different abstract micro descriptions can correspond to what one can say about the same nature, macroscopically.

This work does not move, though, to a post-Bohr reality, in trying to open the "black box" of QM. But we support the Bohr model in a software-defined QC, where $GF(2^m)$ can be used with binary logic to implement Bohr's "many states at once" model, without breaking the LEM in the macro, in universality. This is our "new hypothesis", closing the former four contradictions, where information is given by an abstract, algebraic approach with ternary object symmetry and extensions, modeled by $GF(3^n)$ and implementable as $GF(2^m)$.

In particular, this is important today, when it is wellknown that a shadow has fallen over the race to detect a new type of quantum particle, the Majorana fermion, that could power quantum computers. The Nature retraction is a setback for Microsoft's approach to quantum computing, as researchers continue to search for the exotic quantum states. While the evidence of elusive Majorana particle dies – computing hope lives on, and is now made possible by using tri-state+ in software with standard binary hardware, while enabling the use of spintronic methods and other novel approaches using integers.

5. DISCUSSION

In standard QM, the Schrödinger Equation for bound states is well-known. Eq. (1) is one of only a few solvable models in QM, and shares many qualitative features with physically important models, e.g., tuning of quantum-well lasers by long wavelength radiation, and in the scaling of magnetic fields using Rydberg atoms.

We showed that the double-slit experiment is often wrongly seen, including in cited references. Using low light fields so as to consider only one photon in the apparatus at a time, because only one photon came in, is *not valid* since the number of particles is not conserved, which becomes more important at lower frequencies. We formulate here a consistent QM framework using universality, albeit without any continuity hypothesis. This was clarified by examples in complex analysis.

We showed that it is desirable, therefore, to isolate those aspects of the current QM theory that involve continuous quantities, and are subject to modification by a more satisfactory theory, from aspects that involve only discrete values and are thus relatively more stable, and trustworthy.

This framework reduces to an equivalence of Eq. (1), without using second derivatives, which eschews continuity, and was validated in specifically four major potential models: harmonic, Coulomb, linear, and Rydberg states, at any frequency.

One cannot split the photon at the double-slit experiment, Huygens and all considerations. It would not be one particle anymore. This is not just semantics, this is the semantics, and the ME cannot explain continuity or the coherence term either. Although coherence gives origin to the laser, and stimulated emission. Everything has a stimulated emission component, with in-phase particle creation, even the light from an ordinary candle. But the ME fail to express the stimulated emission component. There are other well-known examples, like diamagnetism and superconductivity, which might seem at first disturbing, where the ME fails but QM explains. However, keeping the ME and Huygens' principle are right and valuable to Physics in universality, in the macro, as considered here. But one cannot use the macro to explain the micro, while the reverse seems possible. As we provided with the "new hypothesis", so we can support the LEM, the ME, and the Huygens' principle, all in the macro, and doing away with the four contradictions. Information, in universality, is given by an abstract, algebraic approach with ternary object symmetry and extensions, modeled by $GF(3^n)$ in the micro and implementable as $GF(2^m)$ in the macro.

[5] The discontinuity is often described to mean that between two points there is a nothing – no objects, no atoms, no molecules, no particles, just nothing, where even the word 'nothing' is maybe too much. as basic elements of a mathematical description.

One can also use universality to support the Bohr model of *QC* in the micro, with the photon as a particle in the micro, breaking the LEM with no conservation of the number of particles, in a software-defined *QC*. Here, $GF(2^m)$ can be used with binary logic to implement “many states at once” in the field $\mathbb{Z}/\mathbb{Z}p$, without breaking the LEM in the macro.

This should all be possible without necessarily using special hardware (e.g. quantum annealing), or incurring in decoherence at all, and wholly designed in software, with today’s binary computers, even with a cell phone. This should provide not only exceptional speed, but also much needed cybersecurity, and a fresh approach with wide new opportunities for anyone.

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Bell-Inequality and Two Slit Experiments: Comparing Misapplication of Classical Probability by Feynman and Bell

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Abstract. We start with the discussion on misapplication of classical probability theory by Feynman in his analysis of the two slit experiment (by following the critical argumentation of Koopman, Ballentine, and the author of this paper). The seed of Feynman's conclusion on the impossibility to apply the classical probabilistic description for the two slit experiment is treatment of conditional probabilities corresponding to different experimental contexts as unconditional ones. Then we move to the Bell type inequalities. Bell applied classical probability theory in the same manner as Feynman and, as can be expected, he also obtained the impossibility statement. In contrast to Feynman, he formulated his no-go statement not in the probabilistic terms, but by appealing to nonlocality. This note can be considered as a part of the author's attempts for getting rid off nonlocality from quantum physics.

Key words: Bell-inequality, two slit experiments, comparing misapplication, Feynman, quantum physics

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Неравенство Белла и интерференционные эксперименты: ошибки Белла и Фейнмана в применении классической вероятности

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Аннотация. Мы начинаем с обсуждения ошибочного применения классической теории вероятностей Фейнманом в его анализе интерференционного эксперимента (следуя критической аргументации Купмана, Баллентина и автора этой статьи). Основой вывода Фейнмана о невозможности применить классическое вероятностное описание для интерференционного эксперимента является трактовка условных вероятностей, соответствующих различным экспериментальным контекстам, как безусловных («абсолютных») вероятностей. Затем мы переходим к неравенствам типа Белла. Белл применял классическую теорию вероятностей таким же образом, как и Фейнман, и, как и следовало ожидать, он также получил утверждение о невозможности ее применения. В отличие от Фейнмана, он сформулировал свое утверждение не в вероятностных терминах, а апеллируя к нелокальности. Эту заметку можно рассматривать как часть попыток автора избавиться от нелокальности в квантовой физике.

Ключевые слова: неравенство Белла, интерференционные эксперименты, Фейнман, квантовая физика

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1. INTRODUCTION

Nowadays quantum nonlocality which was highlighted by Bell [1] (see also, e.g., [2–6]) is the basic component of modern quantum physics (in contrast to "good old quantum physics" of Planck, Einstein, Bohr, Heisenberg, Pauli, Fock, Landau, Blohintzev etc.). However, this nonlocality is apparent and appeal to it in the foundational discussions is really misleading.

Since 1990-s, I tried to get rid off nonlocality from quantum physics. Initially the purely probabilistic reasoning was used – to show that Bell misapplied of the classical probability theory (CP) in his derivation of the famous inequality (e.g., [7–9]; see also Kupczynski [10]). Recently I directly appealed to the formalism of quantum theory by coupling the Bell type inequalities with incompatibility of quantum observables [11–15] (see also Boughn [16]).

This paper is coupled to my previous works on CP-misapplication in Bell's argumentation, especially to recent papers [17; 18], where the role of conditional probability (defined classically with the Bayes formula) was highlighted. The starting point is the critique of Feynman's probabilistic analysis of the two slit experiment [19; 20]. Feynman tried to apply CP to model the experimental output (as well as the prediction by quantum theory) of the two slit experiment. However, he misapplied CP by treating conditional probabilities corresponding to variety of experimental contexts as unconditional so to say absolute probabilities. This led him to the conclusion on incompatibility of CP with quantum theory and experiment . This problem in Feynman's probabilistic reasoning is well known for experts (see e.g. Koopman [21], Ballentine [22], and Khrennikov [7; 9; 23]) and this is the good time to recall about it.

Later Feynman-like reasoning was used by Bell. We analyze parallelism of Feynman and Bell reasoning generating the apparent contradiction between CP and quantum physics. We point that, in contrast to Feynman, Bell overshadowed this conclusion by nonlocality issue.

2. CLASSICAL PROBABILITY THEORY

CP was mathematically formalized by Kolmogorov (1933) [24]. This is the calculus of probability measures, non-negative weight $P(A)$ is assigned to any event A; here $0 \leq P(A) \leq 1$.

The main property of CP is its additivity: if two events A_1, A_2 are disjoint, then the probability of disjunction of these events equals to the sum of probabilities:

$$P(A_1 \vee A_2) = P(A_1) + P(A_2). \tag{1}$$

To make theory mathematically interesting, condition of additivity is extended to countable-additivity.

By using Bayes' formula

$$P(A|C) = P(A \wedge C)/P(C),$$

there is introduced the conditional probability $P(A|C)$: the probability that event A would happen under the condition that event C was happened. We are interested in the situation that event C is selection of a complex of experimental conditions, context C. The conditional probability

$$A \to P_C(A) \equiv P(A|C)$$

is also a probability measure; hence it is also additive:

$$P(A_1 \vee A_2|C) = P(A_1|C) + P(A_2|C), \tag{2}$$

for disjoint events.

Consider now a random variable which represents some observable O, then, for each value α, and any two disjoint events C_1 and C_2, we have, for event $C_{12} = C_1 \vee C_2$,

$$P((O = \alpha) \wedge C_{12}) = P((O = \alpha) \wedge C_1) + P((O = \alpha) \wedge C_2). \tag{3}$$

However, generally

$$P((O = \alpha|C_{12}) = P((O = \alpha|C_1 \vee C_2) \neq \\ \neq P((O = \alpha|C_1) + P(O = \alpha|C_2). \tag{4}$$

So, the calculus of conditional probabilities differs from calculus of "absolute probabilities".

3. PROBABILISTIC ANALYSIS OF THE TWO SLIT EXPERIMENT: FEYNMAN VERSUS KOOPMAN, BALLENTINE, KHRENNIKOV

Here we follow Ballentine [22] (see also Koopman [21] and Khrennikov [7; 9; 23]). The two slit experiment can be described as follows. There is a source, a screen with two slits (labeled s_1 and s_2) in it, and a detector. The detector can be moved to measure the particle count rate at various positions. In this way, an experiment can measure the probability of a particle passing through the slit system and arriving at the point, x. Consider the experimental context C_1 such that only slit s_1 is open; the probability of detection at point x is denoted by P_1. (Point x is fixed, so we omit it form notation.) Consider also another experimental context C_2 such that only slit s_2 is open; the probability of detection at point x is denoted by P_2. Finally, consider the experimental context C_{12} such that both slits are open, the probability of detection is P_{12}.

We note that passage of a particle through slit s_1 and passage through slit s_2 are certainly exclusive events. Hence events of passing through C_1 and C_2 are disjoint and event C_{12} of passing either through C_1 or C_2, when both slits are open, is disjunction of these events.

Hence, one might expect (as Feynman did [19; 20]) from additivity of probability (1) and concretely from (3), that P_{12} should be equal to $P_1 + P_2$. But, as is well known, the experiment statistical data shows that this is not true. Hence, it might be concluded that of probability theory does not hold in quantum mechanics.

Ballentine pointed out [22]: "In fact, the above argument draws its radical conclusion from an incorrect application of probability theory." He had in mind that Feynman misleadingly appealed to the formula (1), instead of (4). To make this issue clearer, we introduce the observable $O \equiv O_x$ representing the clicks of a detector located at the point x and yielding the values 0 (no click) and 1 (click). In this notation, $P_i \equiv P_i(O = +1)$ and $1 - P_i = P_i(O = 0)$, as well as $P_{12} \equiv P_{12}(O = +1)$, $1 - P_{12} = P_{12}(O = 0)$. Feynman's probability fallacy can be written as

$$P_{12}(O = \alpha) \neq P_1(O = \alpha) + P_2(O = \alpha), \quad \alpha = 0{,}1. \tag{5}$$

But, in the CP-framework there is no reason to expect that

$$P(O = \alpha | C_{12}) = P(O = \alpha | C_1) + P_2(O = \alpha | C_2), \quad \alpha = 0{,}1. \tag{6}$$

4. FEYNMAN-LIKE REASONING OF BELL

The Bell experiment can be described as follows. There is a source of pairs of particles and two polarization beam splitters (PBSs) with orientations $\theta = (\theta_A, \theta_B)$; each PBS is coupled to a pair of detectors, $D_A(\pm)$, $D_B(\pm)$. Pairing of detector's outputs for Alice and Bob can be represented as observable O taking vector-values (α, β), where $\alpha, \beta = \pm 1$. Now to create the CHSH-combination of correlations, experimenter has to consider four experimental contexts, corresponding to selection of two Alice's angles θ_{A_1}, θ_{A_2} and two Bob's angles θ_{B_1}, θ_{B_2}. We denote the corresponding contexts as C_{ij}, $i, j = 1, 2$. Then correlations are combined of probabilities $P(O = (\alpha, \beta) | C_{ij})$. We have

$$\begin{aligned} \mathrm{CHSH} = \sum_{i,j=1,2} \sigma_{ij} \Big(\big[P\big(O = (+1,\ +1) C_{i,j}\big) + P\big(O = (-1,\ -1) C_{i,j}\big) \big] - \\ - \big[P\big(O = (+1,\ -1) C_{i,j}\big) + P\big(O = (-1,\ +1) C_{i,j}\big) \big] \Big), \end{aligned} \tag{7}$$

where just one of σ_{ij} equals –1, others equal +1. This consistent context-referring of probabilities prevents derivation of the CHSH inequality,

$$|\mathrm{CHSH}| \leq 2. \tag{8}$$

Therefore it is not surprising that it is violated for the experimental data.

In fact, this is the same probability fallacy – mixing of conditional and unconditional probabilities. To derive (8), one has to consider unconditional probabilities and the corresponding CHSH-combination of correlations.

5. CONCLUDING REMARKS

The violation of the Bell inequality is the hottest topic of the modern foundational debates. They are characterized by the diversity of the mutually contradicting interpretations. In this paper I presented the following interpretation:

Bell as well as Feynman "simply" confused conditional and unconditional probabilities. In this way, they found that CP (and hence Cℓassical *physics which is based on CP) contradicts to quantum theory. In contrast to Feynman, Bell did not formalize his conclusion in the purely probabilistic terms. This led him to coupling the purely probabilistic interplay between conditional and unconditional probabilities to the issue of nonlocality.*

In short, this interpretation can be formulated as follows:

In the framework of multi-contextual experiment (as the Bell type experiments: 4 different contexts for the CHSH test), generally there is no Bell inequality, so there is nothing to violate.

Hence, Bell's appealing to quantum nonlocality is misleading.

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Jordan–Wigner Transformation and Qubits with Nontrivial Exchange Rule

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Abstract. Well-known (spinless) fermionic qubits may need more subtle consideration in comparison with usual (spinful) fermions. Taking into account a model with local fermionic modes, formally only the 'occupied' states $|1\rangle$ could be relevant for antisymmetry with respect to particles interchange, but 'vacuum' state $|0\rangle$ is not. Introduction of exchange rule for such fermionic qubits indexed by some 'positions' may look questionable due to general super-selection principle. However, a consistent algebraic construction of such 'super-indexed' qubits is presented in this work. Considered method has some relation with construction of super-spaces, but it has some differences with standard definition of supersymmety sometimes used for generalizations of qubit model.

Key words: Jordan–Wigner transformation, qubits with nontrivial exchange rule, Quantum Informatics

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Преобразование Жордана–Вигнера и кубиты с нетривиальными перестановочными соотношениями

А.Ю. Власов ©

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Аннотация. Хорошо известные фермионные кубиты (без спина) могут потребовать более тонкого рассмотрения по сравнению с обычными фермионами (со спином). Принимая во внимание модель с локальными фермионными модами, необходимо отметить, что формально только «занятые» состояния $|1\rangle$ могут подчиняться принципу антисимметрии по отношению к перестановкам, но это не относится к «вакуумным» состояниям $|0\rangle$. Введение перестановочных соотношений для таких фермионных кубитов, проиндексированных занимаемыми ими позициями, может вызвать определенные вопросы в связи с так называемыми принципами суперотбора. Несмотря на это, возможно ввести согласованную алгебраическую конструкцию таких правил, представленную в данной работе. Рассмотренные методы имеют определённую аналогию с конструкцией супер-пространств, но при этом обладают некоторыми отличиями от стандартных определений суперсимметрии иногда используемых в обобщенных моделях кубитов.

Ключевые слова: преобразование Жордана–Вигнера, кубиты с нетривиальными перестановочными соотношениями, квантовая информатика

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1. INTRODUCTION

Analogues of fermionic creation and annihilation (ladder) operators were suggested by Richard Feynman for description of quantum computers already in the very first works [1; 2]. However, Jordan–Wigner transformation [3] is necessary to make such operators anticommutting for *different* qubits. Such approach was used later in so-called *fermionic quantum computation* [4].

Representation of fermionic ladder operators in such a way formally requires some consequent indexing (order) for description of Jordan–Wigner transformation. The order does not manifest itself directly in algebraic properties of ladder operators, but transformations of states formally depend on such indexes in rather nonlocal way.

States of physical bosons and fermions can be described in natural way by symmetric and antisymmetric tensors respectively, but fermionic quantum computation is rather relevant with more subtle exchange behavior of some quasi-particles.

Formally, qubits in state $|0\rangle$ corresponds to 'empty modes' and only qubits in state $|1\rangle$ treated as 'occupied modes' could be relevant to fermionic exchange principle for qubits marked by some indexes. Consistent mathematical model of such 'super-indexed' states is suggested in presented work.

More detailed description of such states is introduced in Sec. 2.1 together with formal definition of *signed exchange rule* and 'super-indexed' qubits denoted further as S-*qubits*. The different kinds of operators acting on *S*-qubits are constructed in Sec. 2.2.

Algebraic models of *S*-qubits are discussed in Sec. 3. The 'non-trivial' non-commutative part of such model is similar with exterior algebra recollected in Sec. 3.1. However, 'trivial' commutative elements could not be naturally presented in such a way and more complete model is suggested in Sec. 3.2. The Clifford algebras initially used in Sec. 2.2 for applications to gates and operators become the basic tools here. The *S*-qubits are introduced as minimal left ideals of the Cliford algebras. Finally, some comparison with possible alternative models of qubits related with super-spaces are outlined in Sec. 3.3.

2. SUPER-INDEXED QUBITS

2.1. STATES

Let us introduce special notation for qubits marked by some set of indexes *I* with basic states denoted as

$$\widehat{\left|{}^{a}_{\mu}\right\rangle}\widehat{\left|{}^{b}_{\nu}\right\rangle}\ldots=\widehat{\left|{}^{a}_{\mu},\ {}^{b}_{\nu},\ \ldots\right\rangle},\quad a,\,b,\,\ldots\in\Im,\quad \mu,\,\nu,\,\ldots\in\{0,\,1\}. \quad (1)$$

All indexes in the sequence *a*, *b*, … must be *different*. The $\Im$ can be associated with some nodes in multi-dimensional lattices, more general graphs or other configurations *without natural ordering*. Thus, the qubits in Eq. (1) may be rearranged in different ways.

An idea about basic states of qubits as 'occupation numbers' of anticommuting 'local fermionic modes' (LFM) [4] can be formalized by introduction of equivalence relation between elements Eq. (1) with different order of the indexes defined for any neighboring pair by *signed exchange rule*

$$\widehat{\left|{}^{a}_{0}\right\rangle}\widehat{\left|{}^{b}_{0}\right\rangle}\simeq\widehat{\left|{}^{b}_{0}\right\rangle}\widehat{\left|{}^{a}_{0}\right\rangle},\quad \widehat{\left|{}^{a}_{0}\right\rangle}\widehat{\left|{}^{b}_{1}\right\rangle}\simeq\widehat{\left|{}^{b}_{1}\right\rangle}\widehat{\left|{}^{a}_{0}\right\rangle},\quad \widehat{\left|{}^{a}_{1}\right\rangle}\widehat{\left|{}^{b}_{1}\right\rangle}\simeq-\widehat{\left|{}^{b}_{1}\right\rangle}\widehat{\left|{}^{a}_{1}\right\rangle},$$

i.e.

$$\widehat{\left|{}^{a}_{\mu}\right\rangle}\widehat{\left|{}^{b}_{\nu}\right\rangle}\simeq(-1)^{\mu\nu}\widehat{\left|{}^{b}_{\nu}\right\rangle}\widehat{\left|{}^{a}_{\mu}\right\rangle};\quad \forall a\neq b\in I;\quad \mu,\,\nu\in\{0,\,1\}. \quad (2)$$

Due to such rules terms $\widehat{|0\rangle}$ with 'attached' indexes can be exchanged ('commute') with any state $\widehat{|\psi\rangle}=\alpha\widehat{|0\rangle}+\beta\widehat{|1\rangle}$, but two $\widehat{|1\rangle}$ require change of the sign for such a swap ('anti-commute'). For standard notation and qubits without special indexes the exchange rule could be implemented by *signed swap operator*

$$\hat{S}_{\pm}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1\end{pmatrix}. \quad (3)$$

States Eq. (1) with equivalence relation Eq. (2) define basis of some linear space *S* denoted here as '*super-indexed*' *qubits* or *S-qubits*.

The equivalence relation Eq. (2) can be extended on arbitrary permutation π. Such operator is denoted further $\hat{\pi}_{\pm}$ and notation $\hat{\pi}$ is reserved for usual permutation. The construction of $\hat{\pi}_{\pm}$ does not depend on decomposition of π on adjacent transpositions, i.e., swaps of *S*-qubits considered above. Such consistency becomes more natural from algebraic models below in Sec. 3 and 'physical' interpretation with LFM.

It can be also proved for arbitrary state by direct check for the basis. Let us consider for a given basic state different sequences of transpositions produced *the same permutation*. It is necessary to show that the sign does not depend on the decomposition of the permutation into the sequence. Let us consider restriction (π_1) of permutation on subset of indexes corresponding to *S*-qubits with unit value. For the only nontrivial case the restriction of swap on such subset corresponds to exchange of two units with change of sign. So, for any decomposition the basic vector may change the sign only if the permutation π_1 is odd. Thus, $\hat{\pi}_{\pm}$ is the same for any decomposition of $\hat{\pi}$ on transpositions defined by *signed exchange rule* Eq. (2).

The relation $\hat{S}_{\pm}$ can be considered as a formalization of swap with two LFM denoted as '⇔' in [4]. It could be expressed as composition of usual exchange of qubits '↔' and '*swap defect*' *operator* [4]

$$\hat{D}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}. \quad (4)$$

Perhaps, the term 'fermionic qubits' might be not very justified for the model considered here, because the property Eq. (2) would correspond to fermion for $\widehat{|1\rangle}$ ('occupied,' $n=1$) and boson for $\widehat{|0\rangle}$ ('empty,' $n=0$).

Thus, *S*-qubits could be considered as quasi-particles ('fermisons') with *combined statistics*, because exchange rule instead of (±1) multiplier for bosons or fermions should use *swap defect* operator

$$\widehat{\left|{}^{a}_{\psi}\right\rangle}\widehat{\left|{}^{b}_{\varphi}\right\rangle}\mapsto(-1)^{\hat{n}_a\hat{n}_b}\left(\widehat{\left|{}^{b}_{\varphi}\right\rangle}\widehat{\left|{}^{a}_{\psi}\right\rangle}\right) \quad (5)$$

where a formal representation $\hat{D}=(-1)^{\hat{n}_a\hat{n}_b}$ is used, where $\hat{n}_a$ and $\hat{n}_b$ are analogues of *occupation number operators* defined for usual qubit as

$$\hat{n}|\nu\rangle=\nu|\nu\rangle,\quad \nu\in\{0,\,1\},\quad \hat{n}=\begin{pmatrix}0&0\\0&1\end{pmatrix}. \quad (6)$$

The result of a swap Eq. (5) is defined in simple 'product' form Eq. (2) for the basis, but for arbitrary states the expressions are less trivial.

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The *S*-qubit also could be compared with an element of *super vector space*, but due to some subtleties outlined in Sec. 3.3 such approach should be discussed elsewhere.

The scalar product of *S*-qubits states can be naturally defined for the equivalent sequences of indexes *S* in both ('bra' and 'ket') parts

$$\langle \overset{S}{\Psi} | \overset{S}{\hat{\Phi}} \rangle = \langle \Psi | \Phi \rangle. \quad (7)$$

The definition Eq. (7) does not depend on a sequence *S*. Indeed, let us consider permutations of indexes $\pi: S \mapsto S'$. For the basic states a permutation may only introduce (±1) multiplier and the scalar product Eq. (7) does not change. It can be also checked directly for arbitrary states

$$\langle \overset{S'}{\Psi'} | \overset{S'}{\hat{\Phi}'} \rangle = \langle \Psi | \hat{\pi}_{\pm} \hat{\pi}_{\pm} | \Phi \rangle = \langle \Psi | \Phi \rangle = \langle \overset{S}{\Psi} | \overset{S}{\hat{\Phi}} \rangle. \quad (8)$$

2.2. OPERATORS

Description of quantum gates with *annihilation* and *creation* ('ladder') operators was initially suggested by *R*. Feynman [1; 2]. However, despite of formal resemblance with Pauli exclusion principle for fermions

$$\hat{a}^{\dagger}|0\rangle = |1\rangle; \quad \hat{a}|1\rangle = |0\rangle; \quad \hat{a}|0\rangle = \hat{a}^{\dagger}|1\rangle = 0, \quad (8a)$$

they do not satisfy *canonical anticommutation relation* (CAR) for different qubits. Sometimes, usual qubits are compared with so-called 'hardcore' bosons, but it is not discussed here. It is considered instead, how ladder operators with CAR can be introduced for *S*-qubits, see Eq. (14) below.

The *creation operator* $\mathfrak{a}_a^{\dagger}$ can be defined for basic states taking into account exchange rule Eq. (2) of *S*-qubits

$$\mathfrak{a}_a |\overset{L}{\dots}, \overset{a}{0}, \overset{R}{\dots}\rangle = |\overset{a}{1}, \overset{L}{\dots}, \overset{R}{\dots}\rangle, \quad \mathfrak{a}_a |\overset{L}{\dots}, \overset{a}{1}, \overset{R}{\dots}\rangle = 0, \quad (9)$$

where *L* and *R* correspond to arbitrary sequences before and after index '*a*' respectively. The conjugated *annihilation operator* $\mathfrak{a}_a = (\mathfrak{a}_a^{\dagger})^{\dagger}$ *in* simpler case with index '*a*' in the first position can be written

$$\mathfrak{a}_a \hat{|}\overset{a}{0}, \dots\rangle = 0, \quad \mathfrak{a}_a \hat{|}\overset{a}{1}, \dots\rangle = \hat{|}\overset{a}{0}, \dots\rangle. \quad (10)$$

A sign for application of operator $\mathfrak{a}_a$ to arbitrary position should be found using rearrangement of indexes together with signed exchange rule Eq. (2).

Let us denote $\pm_a$ sign derived from Eq. (2) for expressions such as

$$\hat{|}\overset{L}{\dots}, \overset{a}{1}, \overset{R}{\dots}\rangle = \pm_a \hat{|}\overset{a}{1}, \overset{L}{\dots}, \overset{R}{\dots}\rangle, \quad (11)$$

where $\pm_a = (-1)^{\#L}$ with $\#L = \sum_{l \in L} n_l$ is number of units in sequence *L of* positions before '*a*'. for simplicity *L* and *R* are omitted further.

Finally, Eqs. (9, 10) can be rewritten

$$\mathfrak{a}_a \hat{|}\dots, \overset{a}{0}, \dots\rangle = \pm_a \hat{|}\dots, \overset{a}{1}, \dots\rangle, \quad \mathfrak{a}_a \hat{|}\dots, \overset{a}{1}, \dots\rangle = 0; \quad (12a)$$

$$\mathfrak{a}_a \hat{|}\dots, \overset{a}{0}, \dots\rangle = 0, \quad \mathfrak{a}_a \hat{|}\dots, \overset{a}{1}, \dots\rangle = \pm_a \hat{|}\dots, \overset{a}{0}, \dots\rangle. \quad (12b)$$

For consequent indexes $a = 0, \dots, m-1$ Eq. (12) is in agreement with usual Jordan–Wigner transformation [3; 4], i.e.,

$$\mathfrak{a}_a |n_0, \dots, n_{a-1}, 1, n_{a+1}, \dots\rangle = (-1)^{\sum_{k=0}^{a-1} n_k} |n_0, \dots, n_{a-1}, 0, n_{a+1}, \dots\rangle$$
$$a_a |n_0, \dots, n_{a-1}, 0, n_{a+1}, \dots\rangle = 0, \quad (13a)$$

and $\mathfrak{a}_a^{\dagger}$ is Hermitian conjugation

$$\mathfrak{a}_a |n_0, \dots, n_{a-1}, 0, n_{a+1}, \dots\rangle = (-1)^{\sum_{k=0}^{a-1} n_k} |n_0, \dots, n_{a-1}, 1, n_{a+1}, \dots\rangle$$
$$\mathfrak{a}_a |n_0, \dots, n_{a-1}, 1, n_{a+1}, \dots\rangle = 0. \quad (13b)$$

The $\mathfrak{a}_a$ and $\mathfrak{a}_a^{\dagger}$ defined in such a way satisfy *canonical anticommutation*

$$\{\mathfrak{a}_a, \mathfrak{a}_b\} = \{\mathfrak{a}_a^{\dagger}, \mathfrak{a}_b^{\dagger}\} = 0; \quad \{\mathfrak{a}_a, \mathfrak{a}_b^{\dagger}\} = \delta_{ab}1. \quad (14)$$

Let us now introduce *Clifford algebra* $C\ell(2m)$ with $2m$ generators using operators Eq. (13)

$$\mathfrak{e}_a = i\{\mathfrak{a}_a^{\dagger} + \mathfrak{a}_a\}; \quad \mathfrak{e}'_a = \mathfrak{a}_a - \mathfrak{a}_a^{\dagger}. \quad (15)$$

Operators of so-called *Majorana fermionic modes* coincides with Eq. (15) up to imaginary unit multiplier [4].

With earlier definitions of annihilation and creation operators it can be expressed for basis as

$$\mathfrak{e}_a \hat{|}\dots, \overset{a}{0}, \dots\rangle = \pm_a i \hat{|}\dots, \overset{a}{1}, \dots\rangle, \quad \mathfrak{e}_a \hat{|}\dots, \overset{a}{1}, \dots\rangle = \pm_a i \hat{|}\dots, \overset{a}{0}, \dots\rangle;$$
$$\mathfrak{e}'_a \hat{|}\dots, \overset{a}{0}, \dots\rangle = \mp_a \hat{|}\dots, \overset{a}{1}, \dots\rangle, \quad \mathfrak{e}'_a \hat{|}\dots, \overset{a}{1}, \dots\rangle = \pm_a \hat{|}\dots, \overset{a}{0}, \dots\rangle, \quad (16)$$

where $\mp = -(\pm_a)$.

For consequent indexes $j = 0, \dots, m-1$ the Eq. (16) again correspond to Jordan–Wigner formalism with definition of complex Clifford algebra $C\ell(2m, C)$ by tensor product of Pauli matrices [3; 5]

$$\mathfrak{e}_j = i\underbrace{\hat{\sigma}^z \otimes \cdots \otimes \hat{\sigma}^z}_{j} \otimes \hat{\sigma}^x \otimes \underbrace{\hat{1} \otimes \cdots \otimes \hat{1}}_{m-j-1};$$
$$\mathfrak{e}'_j = i\underbrace{\hat{\sigma}^z \otimes \cdots \otimes \hat{\sigma}^z}_{j} \otimes \hat{\sigma}^y \otimes \underbrace{\hat{1} \otimes \cdots \otimes \hat{1}}_{m-j-1}. \quad (17)$$

However, Eq. (17) directly introduces order of indexes unlike more abstract definitions of operators such as Eq. (12) and Eq. (16) respecting structure of *S*-qubits without necessity of predefined order.

The linear combinations of all possible products with operators $\mathfrak{e}_a$, $\mathfrak{e}'_a$ (or $\mathfrak{a}_a$, $\mathfrak{a}_a^{\dagger}$) for given set $\mathfrak{J}$ with $m_{\mathfrak{J}}$ indexes generate Clifford algebra $C\ell(2m_{\mathfrak{J}}, \mathbb{C})$ with dimension $2^{2m_{\mathfrak{J}}}$. Thus, an arbitrary linear operator on space *S* can be represented in such a way, but unitarity should be also taken into account for construction of quantum gates on *S*-qubits.

An alternative notation $\mathfrak{e}_{a'} = \mathfrak{e}'_a$, $a' \in \mathfrak{J}' \sim \mathfrak{J}$ unifying two sets of generators from Eq. (15) into the single collection with doubled set of indexes $2\mathfrak{J} = \mathfrak{J} \cup \mathfrak{J}'$ is also used further for brevity. Definition of Clifford algebra $C\ell(2m_{\mathfrak{J}}, \mathbb{C})$ can be written with such a set as

$$\{\mathfrak{e}_a, \mathfrak{e}_b\} = -2\sigma_{ab}1, \quad a, b \in 2\mathfrak{J}. \quad (18)$$

and conjugation of elements as

$$\mathfrak{e}_a^{\dagger} = -\mathfrak{e}_a, \quad a \in 2\mathfrak{J}. \quad (19)$$

The elements Eq. (15) generate $C\ell(2m_{\mathfrak{J}}, \mathbb{C})$ isomorphic with whole algebra of $2^m \times 2^m$ complex matrices. The unitary gates may be expressed as exponents of Hermitian elements with pure imaginary multipliers discussed below.

Let us consider for some sequence *L* with *l* indexes from $2\mathfrak{J}$ a product of *l* generators

$$\mathfrak{e}_L = \mathfrak{e}_{a_1} \dots \mathfrak{e}_{a_l}; \quad a_1, \dots, a_l \in 2\mathfrak{J}. \quad (20)$$

Linear subspaces $C\ell^{(l)}$ is introduced as a span of such products, $\mathfrak{e}_L \in C\ell^{(l)}$. The notation $C\ell^0$ and $C\ell^1$ is reserved here for standard decomposition of $C\ell$ as Z_2-graded algebra with two subspaces corresponding to linear span of all possible products with even and odd l respectively [10]

$$C\ell(n) = C\ell^0(n) \otimes C\ell^1(n). \tag{21}$$

The square of element $\mathfrak{e}_L$ can be expressed as

$$\mathfrak{e}_L^2 = (-1)^{\varsigma_l}; \quad \varsigma_l = \frac{l(l+1)}{2} \bmod 2. \tag{22}$$

All such elements are unitary with respect to conjugation operation [5]

$$\mathfrak{e}_L = (-1)^{\varsigma_l}\mathfrak{e}_L; \quad \mathfrak{e}_L\mathfrak{e}_L = 1. \tag{23}$$

The construction of Hermitian basis is also straight forward

$$\left(i^{\varsigma_l}\mathfrak{e}_L\right) = (-i)^{\varsigma_l}\mathfrak{e}_L = (-i)^{\varsigma_l}(-1)^{\varsigma_l}\mathfrak{e}_L = i^{\varsigma_l}\mathfrak{e}_L. \tag{24}$$

An exponential representation of unitary operators is simply derived from Eq. (24) for arbitrary compositions of basic elements, e.g., for $\mathfrak{h}_l \in C\ell^{(l)}$

$$u(\tau) = \exp\left(-\iota_l \mathfrak{h}_l \tau\right), \quad \iota_l = i \cdot i^{\varsigma_l} = i^{\varsigma_l + 1}, \quad \tau \in \mathbb{R}. \tag{25}$$

Due to property $\varsigma_{l+4} = \varsigma_l$ multipliers can be given in the table,

l mod 4	0	1	2	3
ς_l	0	1	1	0
$-\iota_l$	$-i$	1	1	$-i$

(26)

Expression of unitary group $U(2^m)$ using families of quantum gates can be derived using approach with exponents due to correspondence between Lie algebras and Lie groups. The method initially was suggested for construction of universal set of quantum gates [6; 7; 8]. Clifford algebra $C\ell(2m)$ with *Lie bracket operation* defined as a standard commutator

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a} \tag{27}$$

can be used for representation of Lie algebra of special unitary group $su(2^m)$ and group $SU(2^m)$ can be expressed as exponents of elements from $C\ell(2m)$.

In the exponential representation analogue of one-gates for S-qubits with l =1, 2 canbeexpressedas

$$\mathfrak{u}_j = \exp(h_1\mathfrak{e}_j + h_2\mathfrak{e}'_j + h_3\mathfrak{e}_j\mathfrak{e}'_j), \quad h_1, h_2, h_3 \in \mathbb{R}. \tag{28}$$

It can be also rewritten

$$\mathfrak{u}_j = q_0 + q_1\mathfrak{e}_j + q_2\mathfrak{e}'_j + q_3\mathfrak{e}_j\mathfrak{e}'_j; \quad q_0^2 + q_2^2 + q_2^2 + q_3^2 = 1, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}. \tag{29}$$

Analogue of two-gates for S-qubits with indexes $j, k \in \mathfrak{I}$ can be declared by analogue exponents with linear combination of different products including $\mathfrak{e}_j$, $\mathfrak{e}'_j$, $\mathfrak{e}_k$, $\mathfrak{e}'_k$ with coefficients are either real (l = 1, 2) or pure imaginary (l = 3, 4).

Similar exponents with more general elements from linear subspaces $C\ell^{(l)}$ for l = 2 and l = 1, 2 (with arbitrary combinations of indexes from $2\mathfrak{I}$) generate 'non-universal' subgroups isomorphic to Spin $(2m)$ and Spin $(2m + 1)$ respectively [5; 8], but inclusion element with l = 3 is enough to generate unitary group $SU(2^m)$ [8].

Due to physical reasons for some models only terms with *even number of* generators must be used [4]. Formally, such terms belong to even subalgebra $C\ell^0$ that may be again treated as a Clifford algebra due to standard isomorphism $C\ell(n-1) \cong C\ell^0(n)$ [10]

$$C\ell(n-1) \to C\ell^0(n); \quad \mathfrak{e}_L \mapsto \begin{cases} \mathfrak{e}_L, & \mathfrak{e}_L \in C\ell^0(n-1); \\ \mathfrak{e}_L\mathfrak{e}_n, & \mathfrak{e}_L \in C\ell^1(n-1). \end{cases} \tag{30}$$

Thus, a model with even number of generators in Hamiltonians and universal subset of quantum gate with l = 2, 4 [4] is also described by Clifford algebra due to isomorphism $C\ell^0(2m) \cong C\ell(2m-1)$.

3. ALGEBRAIC MODELS OF *S*-QUBITS

3.1 EXTERIOR ALGEBRA

Let us consider a vector space V with basis x_j, $j = 0, \ldots, m-1$. The exterior (Grassmann) algebra is defined

$$\Lambda(V) = \bigoplus_{k=0}^{m} \Lambda^k(V), \tag{31}$$

where $\Lambda^0(V)$ are scalars, $\Lambda^1(V) = V$ are vectors, and $\Lambda^k(V)$, $k > 1$ are antisymmetric k-forms (tensors) with basis

$$x_{j_1} \wedge \cdots \wedge x_{jk}; \quad j_1 < \cdots < j_k \tag{32}$$

where '$\wedge$' denotes antisymmetric (exterior) product $x_j \wedge x_k = -x_k \wedge x_j$, $x \wedge x = 0$, $\forall x \in V$.

The dimension of whole space $\Lambda(V)$ is 2^m and any basic state Eq. (1) of S-qubits could be mapped into $\Lambda(V)$

$$\hat{\left|\overset{j_1}{n_{j_1}}, \ldots, \overset{j_m}{n_{j_m}}\right\rangle} \mapsto \bigwedge_{\substack{j \in \mathfrak{I} \\ n_j = 1}} x_j. \tag{33}$$

Such a method inserts into exterior product only x_j with indexes j satisfying $n_j = 1$. However, Eq. (33) is one-to-one map and arbitrary form $\Omega \in \Lambda(V)$ corresponds to some state $\hat{|\Omega\rangle}$ up to appropriate normalization.

The creation and annihilation operators in such representation correspond to a known construction of Clifford algebra using space of linear transformations on $\Lambda(V)$ [5] and may be expressed for basis Eq. (32)

$$\mathfrak{a}_j\colon \quad x_{j_1} \wedge \cdots \wedge x_{j_k} \mapsto x_j \wedge x_{j_1} \wedge \cdots \wedge x_{j_k}; \tag{34a}$$

$$\mathfrak{a}_j\colon \quad x_{j_1} \wedge \cdots \wedge x_{j_k} \mapsto \sum_{l=1}^{k} (-1)^l x_{j_1} \wedge \cdots \wedge \left(\delta_{j,j_l} 1\right) \wedge \cdots \wedge x_{j_k}, \tag{34b}$$

where 1 is unit of algebra $\Lambda(V)$ and notation $1 \wedge \omega = \omega \wedge 1 = \omega$, $\omega \in \Lambda(V)$ is supposed in Eq. (34). Such operators satisfy Eq. (14) and respect map Eq. (32) due to consistency of Eq. (34a) with Eq. (9) and Eq. (34b) with Eq. (12b).

The generators of complex Clifford algebra $C\ell(2m, \mathbb{C})$ can be expressed with earlier defined *pair* of generators Eq. (15) for each index and for real case elements $\mathfrak{e}'_j$ might be used to produce $C\ell(m, \mathbb{R})$ [5].

The considered representation of S-qubits with exterior algebra $\Lambda(V)$ despite of one-to-one correspondence Eq. (33) for complete basis may be not very convenient for work with 'reduced' expressions such as Eq. (2), because only qubits with state $\hat{|1\rangle}$ map into different x_a, but any sequence of qubits in state $\hat{|0\rangle}$ formally corresponds to unit scalar $1 \in \Lambda^0(V)$. An approach with Clifford algebras discussed next helps to avoid such a problem.

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3.2. CLIFFORD ALGEBRAS AND SPINORS

For Clifford algebra $C\ell = C\ell(2m, \mathbb{C})$ the space of spinors has dimension 2^m and it can be represented as *minimal left ideal* [9]. The left ideal $\ell \subset C\ell$ by definition has a property

$$\mathfrak{c}\iota \in \ell: \quad \forall \iota \in \ell; \quad \mathfrak{c} \in C\ell. \tag{35}$$

By definition, the (nonzero) *minimal* left ideal does not contain any other (nonzero) left ideal.

The notation with single set of indexes $a \in \mathfrak{I}$ and m pairs of generators $\mathfrak{e}_a$ and $\mathfrak{e}'_a$ is again used below. Annihilations and creation operators corresponding Eq. (15) are also useful further

$$\mathfrak{a}_a = \frac{\mathfrak{e}_a + i\mathfrak{e}'_a}{2i}; \quad \mathfrak{a}_a = \frac{\mathfrak{e}_a - i\mathfrak{e}'_a}{2i}. \tag{36}$$

The minimal left ideal ℓ is generated by all possible products with an appropriate element $\ell_\varnothing$

$$\ell = \Big\{\mathfrak{c}\ell_\varnothing : \mathfrak{c} \in C\ell,\ \ell_\varnothing = \prod_{a\in\mathfrak{I}} \ell_0^a\Big\}, \tag{37}$$

where

$$\ell_0^a = \frac{1 + i\mathfrak{e}_a\mathfrak{e}'_a}{2} = \mathfrak{a}_a\mathfrak{a}_a \tag{38}$$

are N commuting projectors $(\ell_0^a)^2 = \ell_0^a$. Due to identity $\ell_0^a = i\mathfrak{e}_a\mathfrak{e}'_a\ell_0^a$ for any index a it can be written

$$\mathfrak{e}'_a\ell_\varnothing = -i\mathfrak{e}_a\ell_\varnothing. \tag{39}$$

Let us apply definition of ℓ Eq. (37) to linear decomposition of $\mathfrak{c}$ on terms with products of generators $\mathfrak{e}_a$ and $\mathfrak{e}'_a$. Any element of ℓ in Eq. (37) may be rewritten as a linear combination of terms without $\mathfrak{e}'_a$ due to Eq. (39). Thus, ℓ has dimension 2^m with products at most m different generators $\mathfrak{e}_a$ on $\ell_\varnothing$ as a basis. Let us also introduce notation

$$\ell_1^a = \mathfrak{a}_a^\dagger\ell_0^a = \mathfrak{a}_a^\dagger \tag{40}$$

then a basis of spinor space ℓ can be rewritten in agreement with Eq. (1)

$$(\ell_\mu^a\ell_\nu^b\cdots)\ell_\varnothing \leftrightarrow \big|\hat{\ }\overset{a}{\mu}, \overset{b}{\nu}, \ldots\big\rangle, \quad a, b, \ldots \in \mathfrak{I}, \quad \mu, \nu, \ldots \in \{0, 1\}, \tag{41}$$

there all indexes a, b, … *must be different*.

In representation Eq. (17) with consequent indexes $a = 0, \ldots, m-1$ the elements ℓ_0^a correspond to $2^m \times 2^m$ diagonal matrices with units and zeros described by equation

$$\ell_0^a \mapsto \underbrace{1\otimes\cdots\otimes 1}_{a} \otimes\, \ell_0 \otimes \underbrace{1\otimes\cdots\otimes 1}_{m-a-1}, \tag{42}$$

where $\ell_0 = |0\rangle\langle 0|$. Therefore, $\ell_\varnothing$ corresponds to a $2^m \times 2^\mp$ diagonal matrix with unit only in the very first position

$$\ell_\varnothing \mapsto \underbrace{\ell_0 \otimes \ldots \otimes \ell_0}_{m} = |\underbrace{0, \ldots, 0}_{m}\rangle\langle\underbrace{0, \ldots, 0}_{m}|. \tag{43}$$

In such a case a product $\mathfrak{c}\ell_\varnothing$ in definition of ideal ℓ Eq. (37) corresponds to a $2^m \times 2^m$ matrix with only nonzero first column. It can be used for representation of a vector with 2^m components.

For arbitrary sequences of indexes from a set $\mathfrak{I}$ an analogue of Eq. (2) also holds

$$\ell_\mu^a\ell_\nu^b = (-1)^{\mu\nu}\ell_\nu^b\ell_\mu^a, \quad a \neq b \in \mathfrak{I}, \quad \mu, \nu \in \{0, 1\}. \tag{44}$$

The *inequality* of indexes a ≠ *b* is essential, because the *super-commutativity* Eq. (44) *does not* hold for a = *b* if μ ≠ ν

$$\ell_0^a\ell_0^a = \ell_0^a; \quad \ell_1^a\ell_1^a = \ell_0^a\ell_1^a = 0; \quad \ell_1^a\ell_0^a = \ell_1^a; \quad \ell_0^a\ell_1^a \neq \ell_1^a\ell_0^a. \tag{45}$$

Anyway, all indexes of *S*-qubits Eq. (1) are different by definition and inequality in Eq. (45) does not affect considered representation Eq. (41).

However, some other expressions for operators or scalar product Eq. (7) may require more general combinations of indexes. It is discussed below.

For arbitrary element $\ell \in \ell$ the operators $\mathfrak{a}_a$, $\mathfrak{a}_a^\dagger$ can be naturally defined via left multiplication

$$\mathfrak{a}_a : \ell \mapsto \mathfrak{a}_a\ell; \quad \mathfrak{a}_a^\dagger : \ell \mapsto \mathfrak{a}_a^\dagger\ell.$$

With respect to map Eq. (41) it corresponds to Eq. (12). Let us check that.

Operators $\mathfrak{a}_a$, $\mathfrak{a}_a^\dagger$ commute with ℓ_0^b and anticommute with ℓ_1^b for a ≠ *b*. For equivalent indexes quite natural expressions follow from definitions

$$\mathfrak{a}_a\ell_0^a = 0; \quad \mathfrak{a}_a\ell_1^a = \ell_0^a; \quad \mathfrak{a}_a^\dagger\ell_0^a = \ell_1^a; \quad \mathfrak{a}_a^\dagger\ell_1^a \neq 0. \tag{46}$$

Let us rewrite map Eq. (41) with shorter notation for basic states *n*

$$\big|\hat{\ }\overset{\mathfrak{I}}{n}\big\rangle = \big|\hat{\ } \overset{j_a}{n_{j_a}}, \overset{j_b}{n_{j_b}}, \ldots\big\rangle \leftrightarrow \ell_n = \prod_{j\in\mathfrak{I}} \ell_{n_j}^j. \tag{41′}$$

It may be also expressed in an alternative form

$$\ell_n = \prod_{j\in\mathfrak{I}} \big(\mathfrak{a}_j\big)^{n_j}\ell_0^j = \Bigg(\prod_{\substack{j\in\mathfrak{I}\\ n_j=1}} \mathfrak{a}_j\Bigg)\ell_\varnothing. \tag{47}$$

resembling Eq. (33) for Grassmann algebra. Due to Eq. (37) products of operators $\mathfrak{a}_a^\dagger$ are mapped by Eq. (47) into elements of left ideal of Clifford algebra, *cf* Eq. (35).

With respect to map Eq. (41) actions of $\mathfrak{a}_a^\dagger$ are in agreement with Eq. (9) and a_a satisfy an analogue of Eq. (10). Thus, operators $\mathfrak{a}_a$, $\mathfrak{a}_a^\dagger$ and their linear combinations $\mathfrak{e}_a$, $\mathfrak{e}'_a$ are corresponding to Eq. (12) and Eq. (16) respectively.

The scalar product Eq. (7) also can be naturally expressed. Let us find conjugations of ℓ_0^a and ℓ_1^a

$$\ell_0^{a\dagger} = \mathfrak{a}_a\mathfrak{a}_a^\dagger = \ell_0^a; \quad \ell_1^{a\dagger} = \mathfrak{a}_a, \tag{48}$$

The equation

$$\ell_j^{a\dagger}\ell_k^a = \delta_{jk}\ell_0^a; \quad j, k = 0, 1 \tag{49}$$

can be checked directly

$$\ell_0^{a\dagger}\ell_0^a = \ell_1^{a\dagger}\ell_1^a = \ell_0^a; \quad \ell_0^{a\dagger}\ell_1^a = \ell_1^{a\dagger}\ell_0^a = 0$$

together with appropriate expressions for products Eq. (41′)

$$\ell_n^\dagger\ell_n = \ell_\varnothing; \quad \ell_n^\dagger\ell_{n'} = 0. \tag{50}$$

Let us also use notation ℓ_Ψ for representation of arbitrary $|\hat{\ }\Psi\rangle$, i.e., linear combinations of basic states. Then scalar product Eq. (7) can be written using Eq. (50)

$$\ell_\Psi^\dagger\ell_\Phi = \big\langle \overset{\mathfrak{I}}{\Psi}\big|\hat{\ }\overset{\mathfrak{I}}{\Phi}\big\rangle\,\ell_\varnothing = \langle\Psi|\hat{\ }\Phi\rangle\ell_\varnothing, \tag{51}$$

where super-index *I* denotes set of indexes used in Eqs. (41), (41′) and it can be dropped, because all indexes are naturally taken into account in such algebraic expressions with appropriate order.

Special notation can be used for 'scalar part' of an element

$$\mathfrak{c} \in C\ell(2m; \mathbb{C}); \quad \mathfrak{c} = c_1 + \cdots; \quad \mathrm{Sc}(\mathfrak{c}) = c. \tag{52}$$

Eq. (51) can be rewritten now to express the scalar product directly

$$\mathrm{Sc}(\ell_\Psi^\dagger\ell_\Phi) = \mathrm{Sc}(\langle\Psi|\hat{\ }\Phi\rangle\ell_\varnothing) = \langle\Psi|\hat{\ }\Phi\rangle\mathrm{Sc}(\ell_\varnothing) = 2^{-m}\langle\Psi|\hat{\ }\Phi\rangle. \tag{53}$$

Let us introduce an analogue of density operator. For pure state it can be defined

$$\wp_\Psi = \ell_\Psi\ell_\Psi^\dagger \tag{54}$$

with natural property

$$\wp_\Psi \ell_\Phi = \ell_\Psi \ell_\Psi^\dagger \ell_\Phi = \ell_\Psi \langle \Psi \hat{|} \Phi \rangle \ell_\varnothing = \langle \Psi \hat{|} \Phi \rangle \ell_\Psi \ell_\varnothing = \langle \Psi \hat{|} \Phi \rangle \ell_\Psi. \quad (55)$$

Arbitrary operators can be expressed using linear combinations with pairs of basic states

$$\hat{|} n' \rangle \langle n \hat{|} \leftrightarrow \wp_{n',n} = \ell_{n'} \ell_n^\dagger; \quad \wp_{n',n} \ell_\Phi = \ell_{n'} \langle n \hat{|} \Phi \rangle \ell_\varnothing = \Phi_n \ell_{n'}. \quad (56)$$

3.3. SUPER VECTOR SPACES

A *super vector space* [11; 12] is $\mathbb{Z}_2$-graded vector space

$$V = V_0 \oplus V_1; \quad 0,\ 1 \in \mathbb{Z}_2 = \frac{\mathbb{Z}}{2\mathbb{Z}}. \quad (57)$$

The complex super vector space is denoted $\mathbb{C}^{d_0|d_1}$, where d_i is dimension of V_i. The elements $v \in V_0$, $p(v) = 0$ and $\omega \in V_1$, $p(\omega) = 1$ are called *even* and *odd* respectively.

The $\mathbb{Z}_2$-graded (super) tensor product for such elements can be defined using *sign rule*

$$v \hat{\otimes} u = (-1)^{p(v)\,p(u)}\, u \hat{\otimes} v \quad (58)$$

Roughly speaking, one *S*-qubit could be compared with element of $\mathbb{C}^{1|1}$, but such approach encounters difficulties for more *S*-qubits. Indeed, $\mathbb{Z}_2$-graded tensor product Eq. (58) in definition Eq. (1) should use *different* copies of initial space. Such approach may be quite natural in definition of $\mathbb{Z}_2$-graded tensor product of algebras and can be used for construction of Clifford algebras [10; 13].

However, it is not quite clear, how to implement similar idea for construction of *S*-qubits from $\mathbb{C}^{1|1}$, because implementation of m *different* copies of *S*-qubit may require to use bigger spaces such as $\mathbb{C}^{m|m}$.

Let us consider basis of $V = \mathbb{C}^{m|m}$: $e_k^0 \in V_0$, $e_k^1 \in V_1$, $k = 0, \ldots, m-1$. States of qubits $\alpha e_k^0 + \beta e_k^1$ belong to *different* 2*D* subspaces of *V* and their tensor product for $k = 0, \ldots, m-1$ is the linear subspace with dimension 2^m *of* the 'whole' tensor product $V^{\otimes m}$ with dimension $(2m)^m$. However, it may look as not very natural choice.

For more trivial cases $\mathbb{C}^{d_0|0}$ and $\mathbb{C}^{0|d_1}$ the tensor product of super vector spaces could be treated as symmetric and antisymmetric tensors respectively, but in such a case all vector spaces in product are usually considered as identical. Similar approach with identical copies of $\mathbb{C}^{d_0|d_1}$ is also quite common in supersymmetry. Thus, superspace is only briefly mentioned here for comparison with other models of *S*-qubits and the term *super-indexed* is used earlier to emphasize the difference with known supersymmetric model of qubits [14].

It should be mentioned also, that in the spinor model of *S*-qubits discussed in Sec. 3.2 the elements ℓ_μ^a formally do not belong to superalgebra. Despite the super-commutation rule is valid for different super-indexes Eq. (44), it can be violated for the same one, Eq. (45).

4. CONCLUSION

Jordan–Wigner transformation maps some operators Eq. (8a) acting 'locally' on *n* qubits (or spin-1/2 systems) into n fermionic creation and annihilation (ladder) operators. The 'nonlocal' construction of such a map supposes introduction of some formal ordering on the set of qubits. Such ordering may be natural for some simple models such as 1*D* chain. All ladder operators in fermionic system are formally equivalent and unnatural order produces technical difficulties for more general models such as multidimensional lattices and more general graphs.

Antisymmetric algebra may be formally used for equal (unordered) description of ladder *operators*, but it does not answer a question about inequality of *states*. To address such a problem in this work was suggested model of 'super-indexed' *S*-qubits. Equivalence relation necessary for agreement with Jordan–Wigner transformation and anti-commutativity of ladder operators is *signed exchange rule* for *S*-qubits Eq. (2).

Algebraic model of *S*-qubits with such property was also discussed. The model uses Clifford algebras and spinors. Such approach is different with analogue constructions more common in supersymmetric models only briefly discussed in subsection above.

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Tri-State+ Communication Symmetry Using the Algebraic Approach

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Abstract. This work uses the algebraic approach to show how we communicate when applying the quantum mechanics (QM) concept of coherence, proposing tri-state+ in quantum computing (QC). In analogy to Einstein's stimulated emission, when explaining the thermal radiation of quantum bodies in communication, this work shows that one can use the classical Information Theory by Shannon (with two, random logical states only, "0" and "1", emulating a relay), and add a coherent third truth value *Z*, as a new process that breaks the Law of the Excluded Middle (LEM). Using a well-known result in topology and projection as a "new hypothesis" here, a higher dimensional state can embed in a lower-dimensional state. This means that any three-valued logic system, breaking the LEM, can be represented in a binary logical system, obeying the LEM. This satisfies QC in behavior, offering multiple states at the same time in $GF(3^m)$, but frees the implementation to use binary logic and LEM. This promises to allow indeterminacy, such as contingency, reference failure, vagueness, majority voting, conditionals, computability, the semantic paradoxes, and many more, to play a role in logic synthesis, with a much better resolution of indeterminate contributions to obtain coherence and help cybersecurity. We establish a link between Einstein's and Shannon's theories in QM, hitherto not reported, and use it to provide a model for QC without relying on external devices (i.e., quantum annealing), or incurring in decoherence. By focusing on adequate software, this could replace the emphasis in QC, from hardware to software.

Key words: QuIC, interconnects, communication, bit, qubit, qutrit, qudit, qtrust, tri-state+, information, algebraic, validation, quantum, classical, meaning, coherence

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1. INTRODUCTION

In the past few decades, the qubit – a two-level quantum-mechanical system – has attracted considerable attention for its mysterious quantum properties [1; 2]. In trying to open the "black box" in the quantum state of a qubit, with further, better, analysis of the interaction process, one can hope to find a "new hypothesis" where the data make a wider causal sense, bringing much higher speed, cybersecurity, and scalability. The development of nanoelectronics devices also needs to involve quantum computing in allowing prediction of the desired structure of matter. Whether we want to use binary logic, ternary logic, or other is still open. Fast Galois field calculations are available in current binary chips, as well as spintronic methods, all with finite integer fields. They are especially useful for processing-in-memory and neural networks. Digital computers can emulate floating point numbers, continuous results, and CDs can play apparently continuous music from discrete files, in any rhythm, better than analogue. Analogue and continuity have become the approximation, while digital has become the true result. Code is the exact result we see everywhere, while analogue has been more and more deprecated. Work on nanoelectronics may require the development of quantum computers with a fundamentally new architecture. One is starting to see the possibility of increasing the logic level of representation. This has led to the proposals of three-level systems (qutrit) [3], four-level systems (qudit) [4; 5], and three or more logic level systems called qtrust. The algebraic approach of this work is illustrated by qtrust, that uses a variable number of logic states, over extended finite integer fields with at least a ternary base in $GF(3^n)$ [6]. The results suggest that qutrits, qudits, and qtrust offer a promising path toward extending the frontier of quantum computersand possibly nanotechnology. Theoretical work [7] suggests that quantum processors based on three-level logic systems, or qutrits, might require fewer resources to build than one based on qubits. A similar result is offered for qudits [4] and qtrust [6]. Logic does not have to be binary, or incomplete. Ambivalence, e.g., is a valid result in a ternary system.

Here, at this very Moscow State University, Setun (Russian: Сетунь) was a three-level logic computer developed in 1958, as well-known. It was arguably the most modern ternary computer, using the symmetric ternary number system and three-valued ternary logic instead of the two-valued binary logic prevalent in other computers. In 1965, a regular binary computer was used to replace it. But in 1970, a new ternary computer architecture, the Setun-70, was developed; it was implemented as a simulation program running on a binary computer.

This demonstrated that while ternary logic may have computational advantages, binary computers seem to be sufficient. We can extend to more physical arguments, in further topics. For example, the first protocol in quantum cryptography was the BB84, which however may not have taken advantage of the full potential of multiple superposition states [8]. Using three-or-more logic is suitable for describing a quantum cryptography protocol which may have a number of advantages compared to the "binary" BB84 protocol. This is to be published elsewhere. A two-day NSF workshop, held Oct. 31 – Nov. 1, 2019, changed the focus to "Quantum Interconnect" (QuIC), leading to a roadmap focusing on components and introducing QuICs, which report was published by a group with Awschalom and including 34 others [2]. In two-state systems given by qubits, Awschalom [2] seem to present special challenges for QuICs.

Comparatively, the current quantum theory of qubits is linked, however, to the classical "bit", following Boolean or classical logic laws, such as the Law of the Excluded Middle (LEM), which carry only two possible values, "0" and "1", to emulate the workings of a relay circuit, and uses a formless "fluid" analogy of classical information, that can only be blocked (relay open), routed or replicated (relay closed).

It is therefore highly desirable to investigate the model of communication, especially in the quantum regime, of an expected quantum communication system, as key to quantum computation (QC), quantum speed, and cybersecurity.[1] In particular, whether an algebraic approach with a three-or-more valued logic in software can satisfy the postulates of quantum mechanics and can be encoded in the standard binary hardware. This also, in reversal of the usual Bohr correspondence principle, which states that the behavior of systems described by the theory of quantum mechanics must reproduce classical physics in the limit of large quantum numbers. Here, the quantum regime dictates the rules of the classical regime, imposing tri-state, and can improve the binary theory of Shannon [9], as considered in Section 5. A shorter version was used in the actual presentation, and is available online at [6].

2. BREAKING THE LEM

The LEM is broken in the double-slit experiment [1] in QM. This is shown experimentally, as confirmed by all experiments to date [1], with such low light intensities so that only one photon would enter the apparatus.

Theoretically, the general state is given by Ψ, as the one-dimensional Schrödinger equation for bound states in QM [10]:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=\left[E-V(x)\right]\psi(x), \tag{1}$$

where E is the energy and $V(x)$ is the potential, with the boundary conditions $\psi(0) = \psi(\infty) = 0$. Here, Ψ is also the *coherent superposition* of the solutions Ψ_a and Ψ_b, where only slit a or slit *b* are open at the same time:

$$\Psi=\frac{1}{\sqrt{2}}\left(\Psi_a+\Psi_b\right). \tag{2}$$

Thus, the behavior of systems described by the Niels Bohr interpretation of QM[2] does *not* reproduce classical physics in the limit of small quantum numbers, although it reproduces for high quantum numbers, being counter-intuitive [1] to our usual observed experience, in those small numbers.

The reason the double-slit experiment is counter-intuitive is because it breaks the LEM.[3] Freedom from LEM would save cost, as discussed in Section 5, by using three or more logical states. And in order to allow a better resolution of indeterminate contributions (such as in 1-out-of-3 voting), more than two states can be used to obtain coherence.

While many are considering a far-future and expensive hardware solution for QC, such as quantum annealing, this work sees 'breaking the LEM' as an opening to a "new hypothesis" here. Freedom from LEM would save cost, as discussed in Section 5, by using three or more logical states. And in order to allow a better resolution of indeterminate contributions (such as in 1-out-of-3 voting), more than two states can be used to obtain coherence.

If the world is quantum or not, anyway Newton's calculus and real numbers lead to infinitesimals and infinities, which were shown by Brillouin [13] to be non physical. No one can physically push spacetime arbitrarily close to zero, no without limit, or exclude particle creation or annihilation at low frequecies. Galois fields and finite differences can be used, however, to build an alternative to conventional calculus, without infinitesimals or the limit concept, so that the conventional approach of analysis need not to be the only approach, for physicists [14]. The derivative and integral formulas, however, remain the same. The new finite difference theory can be perfectly accurate, and yet there is always a space between integers, which represent different points in spacetime.

However, physics is showing that, although non physical, one can keep using infinitesimals and infinities in mathematics, and not change analysis (i.e., calculus) or limits. This is because a higher dimensional state can embed in a lower dimensional state, as well-known in projection. In physics, the universe can have singularities, be quantum at the core, and yet reality should be the consequence of a continuous-looking *universality* [12; 13] – where we observe this through what can only ever be a far-away reference frame. The details of the microscopic, even breaking the LEM, should not be so relevant to the macroscopic behavior and asserting the LEM, in universality. Can the same be affecting communications, that we are not seeing the microscopic?

In 1916–1917, Einstein [15; 16] famously argued that, in addition to the random processes of spontaneous absorption and spontaneous emission in Eq. (1), a third, new, and coherent process of stimulated emission must exist *microscopically* for physical bodies, as a result providing experimental evidence for the quantum, reproducing exactly the experimental studies of the thermal radiation of bodies in quantum communication, and providing the basis for the later invention of the laser (light amplification by simulated emission of radiation).[4] This is the so-called black-body radiation law, macroscopic, and even normal light from a candle, a lamp, or, a radio wave, have a stimulated emission component. This has been extended recently, as well-known, with collective effects, such as superradiance and superabsorption, into 5 states, but with no essentially new process. Following the ternary pattern, one can readily predict that superstimulated emission should also exist, as a collective effect, and this pattern should go further.

[1] How can one prevent a next SolarWinds, Microsoft Exchange, and the Colonial Pipeline cyberattacks?

[2] We do not support the Copenhagen interpretation [9; 10].

[3] One cannot split the photon at the double-slit experiment, notwithstanding Huygens and all classical considerations, such as the Maxwell equations. It would not be one particle anymore [1].

[4] More than 55 000 laser-related patents have been granted in the United States.

This work's "new hypothesis" here, in trying to open the "black box" in the quantum state, with further, better, analysis of the interaction process, that we hope to find, where the data make a wider causal sense. It is $GF(3^n) \Rightarrow GF(2^m)$, for suitable $m > n \in \mathbb{Z}/\mathbb{Z}p$, where p is a prime, meaning that any three-valued logic system, breaking the LEM, can be represented (i.e, embed) in a binary logical system, obeying the LEM. Therefore, Einstein's "stimulated emission" provides coherence in universality, and applies not only to bodies that we must use to transmit and receive information, but also to how we communicate.

For example, an answer stimulates someone to emit a reply in coherence (stimulated emission), or anti-coherence, and it must be in coherence to be effective, and so on. To communicate, we realize in day-to-day experience that one needs not only information, as surprise, but also coherence, as that which both sides know.

We establish a unity, with this hypothesis, between Einstein's [14; 15] and Shannon's [16; 17] theories, hitherto not reported, with plenty of physical examples, both referring to quantum communication processes of bodies, and extend it to be applicable also when bodies are not used, but communication exists.

3. NETWORK CODING

Network coding, originally proposed in 2000 [18], can now be considered for coherent and secure traffic. Between the source and any of the receivers of an end-to-end communication session, one is not only capable of stopping, relaying and replicating data messages, as Shannon considered, but also of coding incoming messages to produce coded outgoing ones, which has been used classically for attacks and, beneficially, for network coding [19], in preventing attacks, and for peer-to-peer content distribution, since it eliminates the need for content reconciliation, and is highly resilient to peer failures.

The fundamental insight of network coding is that information to be transmitted from the source in a session can be *inferred*, by the intended receivers, and does not have to be transmitted verbatim. A similar concept is found in the well-known spread spectrum techniques, and in cybersecurity [20], where anchors can be used to correct the information received from the source using, e.g., majority voting (1-out-of-3).

The significant aspect in QC, as the result of coherent superposition in Eq. (2), is still that the actual message is one selected from a set of possible messages. This is achieved by coherence, whereby the message is qualified. As a consequence, it has the proper semantics [21], e.g., the proper meaning.

From this perspective, one is not rejecting Shannon's IT because it is binary, two-state, obeys the LEM, uses binary computers, or uses the "fluid" model, in implementation, but proposing Shannon's IT as part of implementing a larger quantum theory of IT with three or more states in logic behavior in QC.

4. SHANNON – A MATHEMATICAL THEORY OF COMMUNICATION

In 1948, Claude Shannon published "A Mathematical Theory of Communication" [16]. Communication is defined [22] as the process whereby information is transferred from one point in spacetime, called the source, to another point in spacetime, called the *destination*. Information is what is transferred from *source* to *destination*, if nothing is transferred the information is zero and there is no communication; information can also be seen as surprise, as to what is received. This relates to uncertainty, and information is a measure of uncertainty, which is then related to entropy. The average information is called the source entropy [22].

The once fuzzy concept of "information" was proposed in a precise way, as stated above, to quantify the fundamental unit of classical information, the "bit", and using binary logic, with the LEM being valid.

Previous work has been included, selectively, in the references given so far. However, we do not criticize any of such previous references, that are necessarily wrong when applied to quantum information systems, but say that the symmetries of a binary system, that must use the LEM and binary logic, are insufficient for a suitable quantum communication process.

5. TRI-STATE VERSUS TWO-STATE

This work advances experimentally in binary logic, the observation that, for the same function, computation can be accomplished better even classically, by using three logical states, rather than one can do with binary logic, which necessarily includes the LEM.

This is perhaps surprising but well-known experimentally in complex digital systems [23], that allow designers to separate behavior from implementation at various levels of abstraction, in order to achieve, routinely, million gate chip designs while working with *tri-state*™ using [24], a ternary logic system as in Fig. (2).

Our argument, in "modus ponens", is that, a coherent logic state, building a "coherent channel", should exist also in classical Information Theory, although embedded in a binary logic system, in order to be able to model the communication that must exist, analogous to the experimental fact that a physical state of simulated emission must microscopically exist in quantum communication using physical systems of atoms, molecules, and plasma, as well-known by Einstein [15; 16].

The "coherent channel" provides coherence, as behavior in both cases, surprisingly, even in the classical computing implementation using binary logic, LEM computers.

A natural question is then satisfied, whether three-or-more valued logic systems can be embedded in a binary logical system. The answer is yes, as well-known in topology and projection. Achieving *freedom from the LEM in the behavior while the implementation can obey the LEM*, *and saving cost*. Cobreros et all. [25] and Fedorov et al. [4] have also analysed it, positively. As shown [23], this would work experimentally, but at the expense of performance – in cost, speed, and noise rejection – and scalability.

But this further establishes, in "modus tollens", a physical unity between Einstein's and Shannon's theories in the quantum regime. We can use it to provide a model for QC without relying on external devices (i.e., quantum annealing), or incurring in decoherence [26].

The question is, can a software breaking the LEM, run on LEM hardware? We discuss it positively, and tri-state can then offer many more discriminating channels than binary logic, allowing a much better resolution of indeterminate contributions to obtain coherence, allowing them to be much better discriminated for and filtered by correlation, not just by clipping.

To those who question that tri-state or more would be somehow "illogical" to consider, one notes that, in unpublished notes, before 1910, Charles Sanders Pierce is well-known to have soundly rejected the idea that all propositions must be either True or False, as in Boolean logic, the same as Frege in semantics [21]. Pierce developed well-understood rules where the LEM is not valid, including some truth tables. A modern treatment can be seen in the results by Jones [27], and our works in publication.

The two-state logic levels are given in Fig. (1) in the next page, offering: (1) a low-level state "0" when the lower transistor is on and the upper transistor is off; and (2) a high-level state "1" when the upper transistor is on and the lower transistor is off.

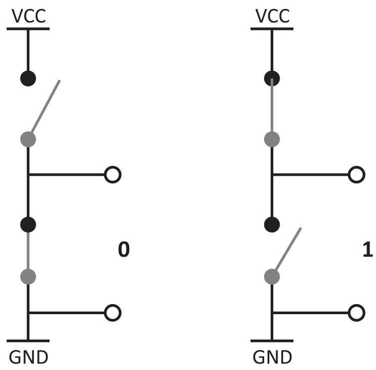

Fig. 1. Example of two-state levels in a circuit, 0 and 1

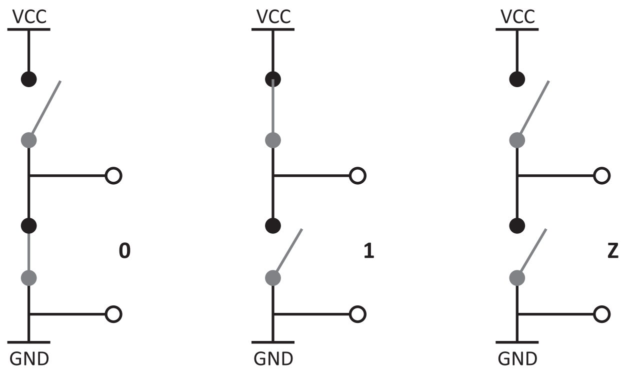

Fig. 2. Example of three states logic: 0, 1, *Z*

To implement three state logic, a physical possibility is a conventional tri-state buffer or gate.[5] This can be seen in Fig. (2), showing the three cases in positive logic:

The solution found for the third logic level, and implemented in [23] devices, was to use a high-impedance state "*Z*", that allows a direct wire connection of many outputs (e.g., routinely with up to a hundred outputs), to a common line, a bus. This exemplifies a programmable (and coherent) interconnect, by the semantics and a challenge-response system, with the ability to move information between different systems that serve distinct tasks at the same time, with *freedom from the LEM in the behavior while the implementation can obey the LEM*, *saving cost*.

Using state *Z* to behave as a coherent interconnect, current information technologies using challenge-response systems, as in a SystemVerilog design [23; 24], can have a semantics to connect to different systems, can avoid race-conditions, handle faults, and maintain a coherent design across different systems. These aspects can also be programmed dynamically at operation time, using tri-state™ designs [23].

The complications of using tri-state logic in implementation of the logic synthesis leads to different benefits/drawbacks in each design, such as using more complex three-valued logic gates or simple two-valued logic gates [23]. As explained above, and as further reviewed in the next Section, "any three-valued logic system, breaking the LEM therefore, can right-represent (i.e, embed) in a suitable two-valued logical system, obeying the LEM."

However, they are not equal, not even entirely equivalent. Any hardware description language (HDL) such as SystemVerilog will eventually be synthesized and different vendors offer different synthesis tools to create devices of their making from the same HDL behavior description. Of interest here, an FPGA vendor, e.g., could code the HDL in a module with a bus they designed. However, when the FPGA is actually synthesized from the code, it would have to use a tri-state buffer because an FPGA cannot output tri-state. Solutions by FPGA vendors such as Actel [28] describe other procedures, since there is no physical tri-state logic inside an FPGA. In this case, an Actel FPGA implements internal multipledrivers on a net with multiplexers instead of three-state logic.

But the states are in different dimensions, and a continuous path in the higher dimension (tri-state) would necessarily map into a discontinuous path in the lower dimension (two-state). This happens due to a well-known theorem in topology and projection, important in communication [22], that we call TR, standing for Topological Reduction. Chiral information (3D), e.g., is well-known not be represented in a projection to 2D, but can in a projection to $GF(2^3)$.

In QC, one can be more precise than physical QM if one makes the model, as the behavior, be more inclusive for coherence, even though implementation should be limited, for practical reasons, to use $GF(2^m)$ and use the LEM. Hence, QC promises to be *easier* to realize than QM.

Three-valued logic, even in $GF(2^m)$ implementations, besides contingency, reference failure, and vagueness, have been associated with at least four other phenomena of interest – namely, conditionals, majority voting, computability, and the semantic paradoxes [25; 29]. These mathematical processes relate to coherence and are inversely related to indeterminacy.

The addition of a third truth value in ternary logic using $GF(3^n)$, or tri-state+, is calculated with *n* in Table 1. Higher *n* orders promise to open the floodgates to a large and near unlimited number of outputs that can be simultaneously considered, for speed and cybersecurity, with many more distinct operators, whatever base one uses. As shown in Table 1, using tri-state+ offers many more discriminating channels, as near 6*e* + 347 possible outputs exist with $n = 3$, for a possible choice of *n*, than $GF(2)$, or binary logic, with two-states, allowing a much better resolution of contributions, with a much improved correlation.

One feels the need to introduce more symmetries than $GF(2)$, or binary logic, in Shannon IT [16]. No longer should we be forced to regard information as a formless "fluid", which can only be blocked, routed, or replicated, obeying the LEM as a "Procrustean bed".[6] For example, one can use majority voting, or 2-out-3 function, even in $GF(2^2)$.

6. THE ALGEBRAIC APPROACH IN QC

This work is based on the algebraic properties of modular arithmetic in number theory, following QC in three or more states, as in $GF(3^n)$ or $GF(2^m)$, and breaking the LEM.

[5] Such as the 74LS241 octal buffer.

[6] Where binary logic, an arbitrary standard, is used to measure success, while completely disregarding obvious harm that results from the effort.

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Modular arithmetic, which is widely known, is a system of arithmetic for finite integers, where numbers "wrap around" when reaching a certain value, called the modulus. One considers two integers x and y to be the same if x and y differ by a multiple of n, and write this as $x = y \bmod n$, and say that x and y are congruent modulo n. Intuitively, division should 'undo multiplication', that is 'to divide' x by y means to find a unique number z such that z times y is x. A unique z exists modulo n only if the greatest common divisor of y and n is 1, and we say that y and n are co-prime.

But while the foregoing is clear, as a subject, Galois fields has also to do with the structure of groups and the relationship with the structure of fields, and how the roots of a polynomial relate to one another. For example, it is easy to implement how finite integer division works, see above, using direct Galois numbers such as $GF(2)$, but it is more involved with extended Galois numbers, such as $GF(2^m)$. Due to universality in physics, seen above, these perhaps confounding factors are of no direct concern here, in communication at a higher level, and are perhaps of interest only to some mathematicians.

In mathematics, a 'field' is any set of elements that satisfies the field axioms for both addition and multiplication and is a commutative division algebra. The group of finite integers modulo p, where p is a prime number, is denoted in mathematics by $\mathbb{Z}/\mathbb{Z}p$. It is well-known that $\mathbb{Z}/\mathbb{Z}p$:

1) is an abelian group under addition;
2) is associative and has an identity element under multiplication;
3) is distributive with respect to addition, under multiplication;
4) is a field.

A mathematical field with a finite number of members is known as a finite field or Galois field. This name is the only property of Galois fields that interest us here.

Finite fields, as $GF(p^n)$, has been useful in the fields of cybersecurity [20], error-correction [30], and encryption, with the well-known AES (Advanced Encryption Standard), where $GF(2^8)$ is used to translate computer data as they are represented in binary, syntactic forms, using Galois extended finite integer fields $GF(2^m)$, with $m = 8$, as well-known.

This work provides for implementation in a binary gate multi-agent environment, while keeping the ternary behavior, and extending it, offering 3, 9, 27... states. It is then possible that other finite integer numbers could be used, and they all would be mathematical fields, but three states seems supported by the formation of the atomic line with "stimulated emission" [14; 15], universality [12; 13] justifies using any number higher than $GF(2)$, breaking the LEM with QM and Eq. (1) requires at least three states, and using Galois fields already extends exponentially any chosen finite integer, as in Table 1, while offering fast hardware support in today's processors with $GF(2^m)$ [30].

With binary logic and diadic operators (2 inputs, binary), there are 16 functions. In electrical engineering, gates implement these functions, notably the AND, OR, NAND, NOR, and XOR (exclusive or) gates. The NAND or XOR gates are functionally complete (meaning that any digital logic circuit can be constructed from either one of these gates) [23]. The XOR function (half adder circuits are implemented with XOR gates) needs brute-force for reversal, and this is basic in cryptographic applications, such as the well-known Advanced Encryption Standard (AES) with $GF(2^8)$. Table 1, summarizes the results, below.

Table 1

Two-state versus $GF(3^n)$

Order	Logic states	Monadic input operators	Dyadic input operators
Binary	2	4	16
1	3	3^3	$3^{3\times 3}$
2	3^2	3^{3^2}	$3^{3^2\times 3^2}$
3	3^3	3^{3^3}	$3^{3^3\times 3^3}$
n	3^n	3^{3^n}	$3^{3^n\times 3^n}$

We see in Table 1, that for diadic functions, the number increases considerably for ternary operators, reaching $3^{3\times 3}$ or 19,683, compared with 16 for binary operators, though not all functionally complete. The diadic ternary functions can help reduce indeterminacy, with a better correlation. There are, clearly, too many functions to enumerate. We will refrain from exploring them, because we can already achieve *freedom from the LEM in the behavior while the implementation can obey the LEM*.

As the number of states in $GF(3^n)$ advance, Table 1 shows that the number of possible operators increase exponentially and, while many are trivial and not functionally independent, a total of near $6e + 347$ diadic operators exists with $n = 3$, to be implemented in QC.

We now remember the statement of the last Section, that "any three-valued logic system, breaking the LEM, can represent (i.e, embed) in a two-valued logical system, obeying the LEM." This calls us to separate behavior from implementation, so that computation, or physical realization, of tri-state+ logic, breaking the LEM, is able to use known binary logic, with LEM components or gate circuits. Proof: the number of binary states in $GF(2^m)$ can increase more than the number of tri-states+ in $GF(3^n)$, with $m > n$.

In other words, three states break the LEM, but $GF(3)$ can be realized in $GF(2^m)$, which obeys the LEM, can use binary functions, and has already more functions than $GF(3)$, for $m = 3$. With ternary logic, the number of monadic functions is 3^3 or 27, while this is exceeded by $GF(2^3)$, with 256 monadic operators. We achieved *freedom from the LEM in the behavior while the implementation can obey the LEM*.

Stimulated emission is seen as a necessary, ternary manifestation of *coherence*, and we propose it (e.g, see our "new hypothesis' in Sections 1 and 6). We call it tri- state+, and it extends itself in a ternary pattern to ever higher orders, captured here by the $GF(3^n)$ symmetries, using an algebraic approach where the number of states is not fixed *a priori*, and coherence effects can be used, further and further, while in communication using $GF(2^m)$ implementations.

Here, therefore, the role of an added mathematical apparatus as discussed here for Galois fields, is not to create unnecessary complications in a description of reality, but implies that there exist more adequate and representative pictures of reality where these other number fields can be used as basic elements of the mathematical description [31].

Accordingly, one moves from the classical Shannon Boolean analogy of circuits with relays, valid for the LEM and a formless and classical "fluid" model of information, with a syntactic expression called 'bit', to a quantum tri- state+, where information is given by an algebraic approach with ternary object symmetry, modeled by $GF(3^n)$ and implementable as $GF(2^m)$.

7. DISCUSSION

By providing a well-known “Procrustean bed”, binary logic shuts off indeterminacy, without processing it, and arrives at a classical, apparently clean and determinate, result that does not take any indeterminacy into account, though reducing design effort.

The properties of such theory have two quantum fatal flaws affecting the “bit” model for QC, in addition to fatal consequences such as the LEM, already disproved in the double-slit experiment in QM. Quantum information is not a formless “fluid”, modeled by a simple object, the “bit”, and that can simply be blocked, routed or replicated; Shannon’s IT is thus not able to take network coding into account. Quantum information further does not always obey binary logic. The Shannon thesis of similitude of communication circuits with relay theory, and with binary logic, thus LEM, is valid only in the classical.

This work’s “new hypothesis”, called tri-state+, in trying to open the “black box” of QM, is then that any three-valued logic system, breaking the LEM, can be represented (i.e, embed) in a suitable binary logical system, obeying the LEM. This agrees with all the theoretical and experimental evidences, dating from 1916, with Einstein and the existence of the quantum. In this “new hypothesis”, where the data can make a wider causal sense with Sannon’s IT, the LEM has limited validity in QC, due to useful indeterminacy contributions, that must remain indeterminate in Eq. (2), the QM solution sought by QC. Thus, coherence effects should be used in communication. This is another example of universality in physics.

A new type of industry, of cybercrime, has been developing profitably also since 2000 [32; 33], which can now be checked by the “new hypothesis”, where coherence effects are used in communication, for cybersecurity[7] not just speed.

This work has discussed a possible new, quantum future, with the ’new hypothesis”. How communication, as a system device or process, should need the quantum symmetries that we call tri-state+, proposing more states than the current binary logic, and without a “Procrustean bed” (i.e. LEM) in behavior.

This suggestion is directly applicable to QC without relying on external devices (i.e., quantum annealing), or incurring in decoherence. Then, QC behavior must break the LEM, as the double-slit experiment in QM already does, and one can implement it today with QM in a binary computer, even a cell phone, and adequate software, in logic synthesis.

To communicate, hence, one needs not only *information*, as surprise, as that which the receiving side ignores, but also *coherence*, as that which both sides know. Building *coherence* is a task that QC can provide with this model, as described, without any special hardware, providing not just speed but cybersecurity.

As another task opened by this work, multilevel logic and mathematics formulas, and software, need to be described and implemented to take full advantage of tri-state+, yet using binary, LEM computers as we know today to implement; this is being published. This could replace the emphasis in *QC*, from hardware to software, saving cost.

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[7] Thus, hoping to prevent another SolarWinds, Microsoft Exchange, and the Colonial Pipeline cyberattacks.

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